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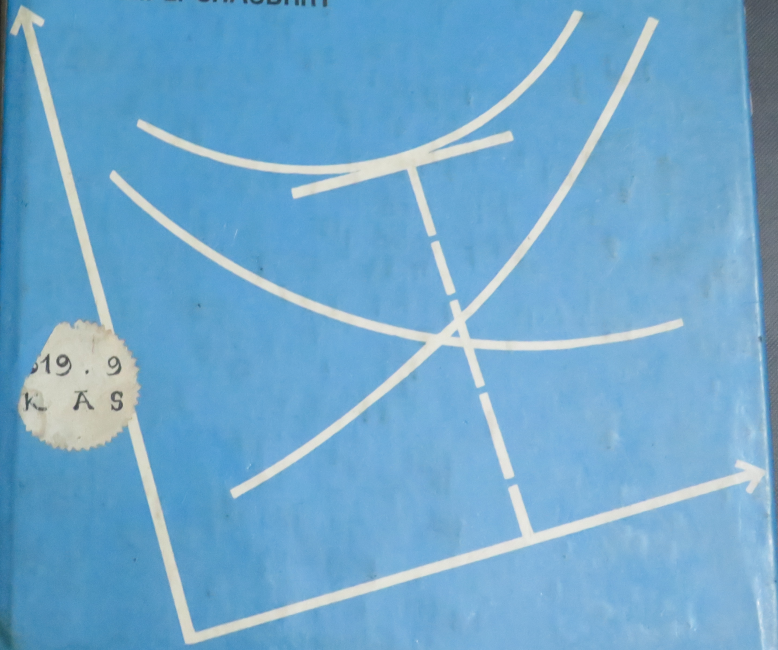
AN INTRODUCTION TO QUEUEING THEORY

B. R. K. KASHYAP
M. L. CHAUDHRY



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ABOUT THE BOOK

This book is primarily intended as a text on queueing theory for students of graduate programmes in management/business administration, mathematics, statistics, computer science and the like. It may also be found useful by advanced undergraduate students and O.R. practitioners. The theory of queues comprising description of various queueing models has been given in the first four chapters, pointing out applications to practical problems form time to time. The last, i.e. the fifth, chapter is exclusively devoted to applications of queueing theory to telephone traffic, inventories, machine interference, scheduling and transportation.

Some special features of the book are :

1. A problem set is provided at the end of each chapter, with answers given for all problems and hints for solution or complete solutions given in several cases.

2. Worked-out examples are provided in the text, wherever found necessary.

3. For most measures like queue length, waiting time, etc., variances are evaluated in addition to means, often with procedures appropriate for evaluation of higher moments.

4. Almost throughout the text, explicit results have been obtained after inversion of Laplace transforms or generating functions.

5. Preliminaries and other relevant data are given in appendices to make the book self-contained for a reader with an undergraduate mathematics background.

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An Introduction to Queueing Theory

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To
Our Teachers
who taught us most of what we know
and to
Our Brothers, **S. P. Kashyap** and **H. R. Chaudhry**
who inspired our learing in many ways

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PREFACE

The present book is primarily intended to serve as a text for graduate students studying operations research in institutes/schools of management/business administration or of technology/engineering or in allied departments like mathematics, statistics and computer science of universities/colleges. It may be found useful by advanced undergraduate students as well as by O.R. practitioners, who need a greater exposure to queueing models than that available in texts on operations research containing a chapter or two on the subject.

The book has grown out of the lecture notes used by the authors for about two decades, modified from time to time and enriched by some of their recent research findings as well as those of contemporary authors. Our aim has been to present the subject matter in a simple, lucid style, devoid of technical jargon as far as practicable. Our thoughts on the subject and our notes have drawn considerably from the existing literature on queueing theory, especially from the books written by authors like Cox and Smith, Morse, Prabhu, Saaty, and Sasieni, Yaspan and Friedman, and some more recent books including those by Chaudhry and Templeton, Gross and Harris, Kleinrock, and Neuts, and to them, among others, we acknowledge our sincere gratitude. Of course, such a list can never have the semblance of being exhaustive.

The book consists of five chapters. Chapter 1 contains a brief introduction to the subject and goes on to the study of steady-state Poisson queues. Starting with the M/M/1 queueing system, it includes various other Poisson queueing models like M/M/1/N, M/M/c, M/M/∞, studied through the use of state-dependent parameters. Chapter 2 includes the transient solution of the M/M/1 queue and the analogous derivation of the busy period distribution. In Chapter 3 we study steady-state non-Poisson queues like M/G/1, M/D/c, GI/M/1, and GI/M/c, using the imbedded Markov chain method and append a continuous-time analysis of the M/G/1 system, using the supplementary-variable technique. Also included are the continuous-time study of the $M/E_k/1$ system as as well as results for M/HE/1 as a special case of the M/G/1 queue. The chapter concludes with a matrix-geometric analysis of the $GI/E_m/1$ system. In Chapter 4 we include miscellaneous topics like the bulk-arrival queueing system $M^x/M/1$, the bulk-service model $M/M^b/1$, queues in

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series and priority queues. For simplicity, only the case of non-preemptive priorities is analyzed. Chapter 5 includes a discussion of practical applications of queueing theory to areas like telephone traffic, inventories, machine interference, scheduling and transportation to serve as an illustrative set of areas of queueing applications, though, really speaking, queueing and congestion problems arise almost in every walk of life.

Finally, as complement to the main text, six appendices have been added. As, for various reasons, many queueing problems are often solved through simulation, we thought it proper to provide a concise introduction to simulation and its application to queues and this has been done in Appendix A. Necessary information on probability as well as on the Laplace and Z-transforms is given in Appendix B, whereas Appendix C contains miscellaneous topics like Rouché's theorem and some not-so-common topics from differential and difference equations, and inversion of transforms. Appendices D and E provide a list of the symbols used and a table of Laplace transforms, respectively. Appendix F is a table of random digits for handy use in simulation.

Some words regarding the special features and the *raison d'être* of this book will not be out of place here, especially when a rich literature on the subject is already available. Some of the special characteristics of the book are the following.

1. All chapters contain Problem Sets at the end. Answers to all problems are given and, in some cases, the problems are completely worked out.

2. Wherever considered necessary, suitable solved examples have been included in the text to give the reader an insight into problem solution.

3. Often the mean is found to be a crude indicator of an efficiency measure of a system. For most of the measures like the queue length, waiting time, etc., we have evaluated the variance as well, sometimes with a procedure given for evaluating higher moments.

4. For almost all topics included in the text, complete solutions are given, including the inversion of the Laplace transforms or generating functions, so that the results are available for use in practical applications.

6. To sum up, though the book is expected to make an interesting reading for any one who has gone through undergraduate mathematics and/or statistics courses, we believe that, with the simple language used and with inclusion of topics in Appendices B and C, it can also be used fruitfully by the not-so-mathematically-oriented students of business, and practising managers.

To quote Francis Bacon, "Studies serve for delight, for ornament, and for ability." We believe this book will be found useful on the three counts.

We are very grateful to Professor J. G. C. Templeton of the University of Toronto for having kindly agreed to go through the entire text and offering suggestions for improvement.

Members of our families have helped us in many ways during the writing of this book, and to them—Raj, Rakesh, Meenakshi and Rajesh Kashyap, and Parphulat, Vandana, Amit and Ajit Chaudhry—we owe many thanks.

Last, but not least, we thank the printers, Eka Press of the Indian Statistical Institute, Calcutta for their excellent work.

We will welcome the readers' comments and suggestions for improvement of the book.

B. R. K. Kashyap
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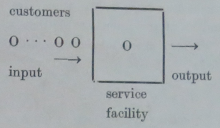
CHAPTER 1

POISSON QUEUES I

1.1 Introduction

In queueing theory we study situations where *units* of some kind *arrive* at a *service facility* for receiving *service* of some description, some of the units having to *wait* for service, and *depart* after service. A *queue* or *waiting line* develops whenever the service facility cannot cope with the number of units requiring service. The units calling for service are often called *customers* in a generic sense. A *system* is generally defined as something having an input, an output and a transformation process in between, which changes the input into the output. Thus, the arrangement described above qualifies to be regarded as a *queueing system*, where the customers requiring service form the input, the serviced customers the output and the service rendered is the transformation process.



The subject of queueing theory had its origin in the pioneering work done by Agner Krarup Erlang, an engineer at the Copenhagen Telephone Exchange around the beginning of this century on the application of probability theory to telephone traffic problems. It soon drew attention of many other probability theorists and has remained a popular field of research almost throughout the eighty years since then. There are many situations in real life where queues of the type described above develop. Thus, we can have a queue of broken-down machines waiting for repair at a repair shop, a queue

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of customers at a departmental store cash counter, or such dangerous queues as the one formed by planes circling above an airport waiting to land. Often we have cases where a physical queue is absent, such as the waiting list of passengers for a railway or airline ticket or of persons who register their names, say, for the purchase of a car, which is not readily available and is to be supplied from future production.

1.2 Characterization

A queueing system is specified by stating the following details about it.

(a) *The input or the arrival pattern.* Let the successive arrivals to the system occur at times $t_1, t_2, \ldots$; then $u_r = t_{r+1} - t_r$ $(r = 1, 2, \ldots)$ are the inter-arrival times. We assume that the u_r are independent and identically distributed (i.i.d.) random variables (r.v.'s) and the input process is specified by stating their probability distribution function (P.D.F.), $A(u)$. $A(u)$ is called the inter-arrival time distribution or, simply, the arrival distribution or the input distribution. Such information as whether arrivals occur singly or in groups may be added here.

(b) *The queue discipline.* It is the statement of the rule of selection of customers for service out of those waiting, and may include rules regarding formation of queues.

(c) *The service mechanism.* It is stated by specifying the number of servers $c \geqslant 1$ and the probability distribution function $B(v)$ of the service times of the successive customers $v_1, v_2, \ldots$, which are assumed to be i.i.d. r.v.'s and also independent of the inter-arrival times u_r $(r = 1, 2, \ldots)$. We may also state whether the service is rendered to customers singly or in groups of fixed or varying size.*

1.3 Topics of Study

In any analysis of a queueing system, one or more of the following aspects of it are investigated.

(a) *The queue length.* This means the number of customers waiting before service. Sometimes the term is used to mean the total number of customers present in the system, including the one or more customers who are being served at the moment. Generally, the precise meaning is clear from the context. However, some authors prefer to avoid confusion by using the term queue length in the former sense, *i.e.* to mean the queue proper (before

* Some recent studies on queues with correlated arrivals or service times have been made.

service), and by calling it the number in the system or the system size when those being served are intended to be included. We shall be following this arrangement.

(b) *The waiting time.* This means the time spent by a customer in waiting before service. This term, too, is used in a wider sense by some authors to mean the time spent in waiting or being served, and they will call the time spent in waiting before service the queueing time. We shall prefer the former meaning, which seems to be more popular, and we shall call the waiting time in the wider sense the total time spent in the system, which is clearly the waiting time (before service) *plus* service time.

(c) *The busy period.* By a busy period is meant a period of time over which the server is continuously busy. Clearly, a busy period starts with an arrival at an empty service facility and ends when the number of customers in the system drops to zero for the first time. At this instant an *idle period* starts, which ends at the commencement of the next busy period.

The queue length, the waiting time and the busy period are studied through their probability distributions, from which moments like mean, variance, etc. can be obtained.

1.4 Notation

We shall use Kendall's (1953) notation, explained below, to denote the various queueing systems to be discussed. Occasionally, when necessary, we shall enhance the notation following the extension of Kendall's notation suggested by Lee (1966).

A queueing system will be denoted by a triad $\cdot/\cdot/\cdot$, in which the first two members will be letters which stand for the forms of the input and the service time distributions, respectively, and the third member is a number signifying the number of servers. The forms of the arrival and service distributions commonly used are described below.

(a) *M (Markovian or the negative exponential distribution).* This is the most commonly used distribution in queueing literature for reasons to be discussed later. When the letter M is used in the first place in the triad mentioned above, it will be taken to mean that the input distribution is given by

$$A(u) = 1 - e^{-\lambda u}, \quad u \geqslant 0 \tag{1}$$

$$= 0, \qquad \text{otherwise}$$

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or, that the corresponding inter-arrival time density function is given by

$$a(u) = \lambda e^{-\lambda u},\ u \geqslant 0 \tag{2}$$
$$= 0, \qquad \text{otherwise}$$

The mean of the above distribution is $1/\lambda$, which is, therefore, the mean inter-arrival time. It follows that λ is the mean arrival rate, *i.e.* the average number of arrivals per unit time. From the well-known relation between the (negative) exponential and the Poisson distributions, it can be seen that the number of arrivals in a given time t will have the following Poisson distribution :

$$\text{P}\{n \text{ arrivals in time } t\} = (\lambda t)^n e^{-\lambda t}/n!,\ n = 0, 1, 2, \ldots \tag{3}$$

Because of the above, the case of exponential inter-arrival times is equivalently stated as Poisson arrivals or Poisson input.

In the same way, when M stands at the second place in the triad, it will mean that the service time distribution is

$$B(v) = 1 - e^{-\mu v},\ v \geqslant 0 \tag{4}$$
$$= 0, \qquad \text{otherwise}$$

the corresponding density function being given as before, with λ, u in (2) replaced by μ, v. As before, we see that $1/\mu$ is the mean service time and μ the mean service rate, and

$$\text{P}\{n \text{ services in time } t\} = (\mu t)^n e^{-\mu t}/n!,\ n = 0, 1, 2, \ldots \tag{5}$$

Note : *Special features of the negative exponential distribution.* The exponential distribution possesses the following two properties, which make it very convenient for use in the analysis of queueing systems.

(i) *The Markovian or the "forgetfulness" property.* The conditional probability of an arrival during the time interval $(u, u+du)$, given that an arrival has not taken place in $[0, u_0]$, is equal to

$$\text{P}\{u < u_r < u+du \mid u_r > u_0\} = a(u)du/[1-A(u_0)] \tag{6}$$
$$= \lambda e^{-\lambda(u-u_0)}\, du$$

which shows that the entire inter-arrival time and the *residual* inter-arrival time after time u_0 have the same exponential distribution. Taking $u_0 = u$, we see that the *probability of an arrival during* $(u, u+du)$ *does not depend upon the time since the last arrival.* That is why Poisson arrivals are also called *random arrivals.*

Similarly, in the case of exponential service times the residual service time and the entire service time have the same distribution and the probability of a service completion is independent of the elapsed service time. Due to this fact, the exponential distribution is not suitable for service times. On the other hand, the exponential distribution is often a good fit for inter-arrival times, especially where arrivals are due to chance phenomena like machine failure, or come from an unstructured system like bus arrivals at a bus stop far from the starting point. In many instances 'customers' arriving at a business shop exhibit such random arrival pattern. There is a saying in Hindi which means that a customer and death have no fixed time.

(ii) In the case of negative-exponentially distributed inter-arrival times, since the arrivals in a given interval follow the Poisson distribution, we have

$$\text{P}\{\text{one arrival during } (t, t+\Delta t)\} = (\lambda\Delta t)e^{-\lambda\Delta t}/1! \tag{7}$$
$$= \lambda\Delta t\{1-\lambda\Delta t+\ldots\}$$
$$= \lambda\Delta t+o(\Delta t)$$

where $o(\Delta t)$ stands for terms which are negligible in comparison to Δt, in the sense that

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0 \tag{8}$$

Similarly,

$$\text{P}\{\text{no arrival during } (t, t+\Delta t)\} = (\lambda\Delta t)^0 e^{-\lambda\Delta t}/0! \tag{9}$$
$$= 1-\lambda\Delta t+o(\Delta t),$$

$$\text{P}\{\text{two or more arrivals during } (t, t+\Delta t)\} = o(\Delta t) \tag{10}$$

In the same way, if service times are also exponentially distributed, we have

$$\text{P}\{\text{one service completion during } (t, t+\Delta t)\} = \mu\Delta t+o(\Delta t) \tag{11}$$

$$\text{P}\{\text{no service completion during } (t, t+\Delta t)\} = 1-\mu\Delta t+o(\Delta t) \tag{12}$$

$$\text{P}\{\text{two or more service completions during } (t, t+\Delta t)\} = o(\Delta t) \tag{13}$$

$$\text{P}\{\text{no arrival and no service during } (t, t+\Delta t)\} = 1-(\lambda+\mu)\Delta t+o(\Delta t) \tag{14}$$

$$\text{P}\{\text{one arrival and one service during } (t, t+\Delta t)\} = o(\Delta t) \tag{15}$$

(b) *D (deterministic or constant inter-arrival/service times).* The letter D in the first (second) place in the triad in Kendall's notation indicates that the inter-arrival time (service time) is constant. In the case of constant

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inter-arrival times, also called regular arrivals or regular input, when each inter-arrival time is a, say, we have the input distribution

$$A(u) = 0, \quad u < a$$
$$= 1, \quad u \geqslant a \quad (0 < a < \infty) \tag{16}$$

and the density function

$$a(u) = \delta(u-a), \tag{17}$$

where $\delta(\cdot)$ is the Dirac delta function. When service times are constant, we have similar expressions for $B(v)$ and $b(v)$.

(c) E_k (*k-Erlang distribution*). We say that the inter-arrival times have a k-Erlang distribution, or an Erlang distribution of order k, if each inter-arrival time is made up of k phases, the time spent in each phase having a negative exponential distribution with a common mean $1/(k\lambda)$, say, independently of other phases, so that the mean inter-arrival time is $1/\lambda$. Thus, the inter-arrival time density $a(u)$ is given by

$$\begin{aligned} a(u)du &= \mathrm{P}\{u < u_r < u+du\} \\ &= \mathrm{P}\{(k-1) \text{ phases are completed in time } u \text{ and} \\ &\qquad \text{one phase in } (u, u+du)\} \\ &= \frac{(k\lambda u)^{k-1}e^{-k\lambda u}}{(k-1)\,!} \cdot k\lambda du \\ &= \frac{(k\lambda)^k u^{k-1}e^{-k\lambda u}}{(k-1)\,!}\, du. \end{aligned} \tag{18}$$

This is clearly the k-fold convolution of the exponential distribution with itself and, therefore, gives the exponential distribution for $k = 1$. Also, it can be seen from (18), or by using the convolution theorem, that the Laplace transform of $a(u)$ is

$$\int_0^\infty e^{-su}a(u)du = \left(\frac{k\lambda}{k\lambda+s}\right)^k = \left(1+\frac{s}{k\lambda}\right)^{-k} \to e^{-s/\lambda} \tag{19}$$

as $k \to \infty$. Now, since the Laplace transform of the density function of a r.v. X is the expectation $E(e^{-s\overline{X}})$, we see that $e^{-s/\lambda}$ corresponds to the case of constant inter-arrival times $= 1/\lambda$, the mean value. So, for $k \to \infty$, the Erlangian input becomes the regular input, given by (16) or (17).

The case of k-Erlang service time distribution can be dealt with similarly, the density function $b(v)$ being given by (18) with λ, u changed to μ, v, respectively.

(d) G (*general distribution*). The letter G is used for a general or arbitrary distribution. However, following the convention, we shall write GI (standing for "general independent") for general inter-arrival times and G for general service time distribution.

From the notation explained above, we see that

(i) M/M/1 stands for Poisson arrivals, exponential service times and one server,

(ii) M/D/c stands for Poisson arrivals, constant service times and c servers,

(iii) M/E$_k$/2 stands for Poisson arrivals, k-Erlang service times and two servers,

(iv) GI/G/c stands for the most general case of general input, general service times and c servers.

Lee (1966) has extended the above notation to $\cdot/\cdot/\cdot:(\cdot/\cdot)$, where the triad is as explained earlier, and within the parentheses the first member is a number, showing the maximum number of units that can be accommodated in the system (queue *plus* service) and the second member is an abbreviation for the queue discipline, like FIFO, LIFO, SIRO, etc., explained below. We shall now describe some of the queue disciplines encountered.

Queue discipline. The most commonly practised queue discipline is "first come, first served" or, as it is sometimes called, "first in, first out", abbreviated, respectively, as FCFS or FIFO. In this case, the waiting customers are selected for service in the order of their arrivals. Various other possibilities are there. Sometimes we have just the opposite rule of "last come, first served" or "last in, first out", in short LCFS or LIFO. In some cases we can have a random selection for service, or "service in random order", briefly, SIRO. And there are also a number of priority disciplines, according to which the customers are divided into two or more priority classes and a higher priority unit is chosen for service before a lower priority unit. Further, under a "preemptive priority" rule, a lower priority unit is preempted or taken out of service whenever a higher priority unit arrives during its service, the preempted lower priority unit being attended to only after serving all the units of higher priority classes. Under "non-preemptive priority" or the "head-of-the-line priority" rule, a service once started is allowed to continue up to completion, irrespective of arrivals of higher priority units, which are attended to only after completion of the service in

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progress. There are also "dynamic priority" disciplines, where the priority increases due to waiting. For a detailed account of priority disciplines in queues, one may refer to Jaiswal (1968).

It may be noted that out of the three queue characteristics of the queue length, the busy period and the waiting time, only the waiting time is affected by the queue discipline.

Impatient customers. In addition to the queue discipline, some traits of behaviour of waiting customers have often to be taken into account. Thus, a customer is said to "balk" if, seeing the size of the queue, he decides not to join the queue and is thus lost to the system. If after waiting for some time, he gets impatient and leaves the system before his service, he is said to "renege".

Systems of queues. In a multi-server queueing system like M/M/c, the *servers* are operating *in parallel* and are fed by a single queue. Alternatively, we may have a separate queue for each server, *i.e.* a number of *queues in parallel*. In such a case, the customers exhibit the phenomenon of shifting between queues time and again, for example, to be always in the shortest queue. This is called *jockeying*. We have an arrangement of *queues in series* or *queues in tandem* when the output of one queue becomes the input to the next queue and so on until the last queue in the series. We have a case of *cyclic queues* if the output of the last queue again becomes input to the first queue. Sometimes we have *queueing networks* consisting of queues arranged both in series and in parallel.

1.5 The Queueing System M/M/1

(a) *The queue length*

Let us consider the single-server queue with Poisson arrivals and exponential service times. Let us write

(1) $$P_n(t) = P\{\text{there are } n \text{ units in the system at time } t\}$$

So, using eq. **1.4** (7)–(15), we have, for $n \geqslant 1$,

(2) $$\begin{aligned} P_n(t+\Delta t) &= P_n(t)P\{\text{no arrival and no service in } (t, t+\Delta t)\} \\ &\quad + P_{n-1}(t)P\{\text{one arrival in } (t, t+\Delta t)\} \\ &\quad + P_{n+1}(t)P\{\text{one service in } (t, t+\Delta t)\} + o(\Delta t) \\ &= P_n(t)\{1-(\lambda+\mu)\Delta t\} + P_{n-1}(t)\cdot\lambda\Delta t + P_{n+1}(t)\cdot\mu\Delta t + o(\Delta t) \end{aligned}$$

whence

(3) $$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{o(\Delta t)}{\Delta t}$$

Proceeding to limits as $\Delta t \to 0$, we have

(4) $$\frac{d}{dt}P_n(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n = 1, 2, \ldots$$

For $n = 0$, we shall have

(5) $$\begin{aligned} P_0(t+\Delta t) &= P_0(t)P\{\text{no arrival in } (t, t+\Delta t)\} \\ &\quad + P_1(t)P\{\text{one service in } (t, t+\Delta t)\} + o(\Delta t) \\ &= P_0(t)\{1-\lambda\Delta t\} + P_1(t)\cdot\mu\Delta t + o(\Delta t) \end{aligned}$$

whence, as before,

(6) $$\frac{d}{dt}P_0(t) = -\lambda P_0(t) + \mu P_1(t).$$

Eq. (4) and (6) are differential-difference equations, which can be solved to evaluate $P_n(t)$, thus giving the time-dependent probability distribution of the number in the system. This solution is quite involved and will be discussed in the next chapter. However, for most of practical applications of the queueing model, a "steady state" solution suffices. We say, in general, that a system has reached a steady state or a state of "statistical equilibrium" when fluctuations with respect to time can be disregarded. Hence, to obtain a steady state solution we shall regard the $P_n(t)$, $n = 0, 1, 2, \ldots$ as independent of time. Consequently we have $dP_n(t)/dt = 0$, and we shall drop the argument t in $P_n(t)$ and write simply P_n. Thus, (4) and (6) yield

(7) $$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1}, \quad n = 1, 2, \ldots$$

(8) $$0 = -\lambda P_0 + \mu P_1.$$

Eq. (7) and (8) are easy to solve recursively, though there are other methods to solve them. (For details see Section 1.7.) It is customary to write

(9) $$(\lambda/\mu) = \rho$$

and call this ratio the "traffic intensity" of the system. From (8), we have

(10) $$P_1 = \rho P_0$$

Now, writing eq. (7) for $n = 1$, substituting for P_1, and solving the resulting equation for P_2, we have

(11) $$P_2 = \rho^2 P_0$$

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Similarly, taking eq. (7) for $n = 2$ and substituting for P_2 and P_1, we have

(12) $$P_3 = \rho^3 P_0$$

and so on for P_4, etc.

It is easy to show by mathematical induction that

(13) $$P_n = \rho^n P_0, \quad n = 1, 2, \ldots$$

Hence, the steady-state probability distribution P_n is known in terms of P_0. To evaluate P_0, we use the fact that the total probability over the entire distribution is one. Thus, if we have no upper limit on the number of customers that can join the queue, we have

(14) $$\sum_{n=0}^{\infty} P_n = 1$$

which is called the normalizing equation.

From (13) and (14), we have

(15) $$P_0(1+\rho+\rho^2+\ldots) = 1$$

Now, the infinite geometric series $1+\rho+\rho^2+\ldots$ is convergent only when $\rho < 1$. We conclude that the steady state exists only when $\rho < 1$. Assuming that $\rho < 1$ we have, from (15),

(16) $$P_0 = 1-\rho$$

so that, from (13),

(17) $$P_n = (1-\rho)\rho^n, \quad (\rho < 1, n = 0, 1, 2, \ldots)$$

which is a geometric probability distribution.

The mean number in the system, which we shall denote by L, is given by

(18) $$\begin{aligned} L &= \sum_{n=0}^{\infty} nP_n \\ &= \sum_{n=0}^{\infty} \{n.(1-\rho)\rho^n\} \\ &= (1-\rho)(\rho+2\rho^2+3\rho^3+\ldots) \\ &= \rho(1-\rho)(1-\rho)^{-2} \end{aligned}$$

Thus,

(19) $$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$

The mean number in the queue (waiting for service), denoted by L_q, is similarly given by

(20) $$\begin{aligned} L_q &= \sum_{n=1}^{\infty} (n-1)P_n \\ &= \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)} \end{aligned}$$

(b) *The waiting time*

To study the waiting time, let us assume that the queue discipline is "first come, first served". Then, the mean waiting time in the queue and in the system, denoted by W_q and W, respectively can be found from simple arguments.

Let us first take W_q. The customers found in the queue at any time are those that arrived during the waiting time of the customer at the head of the line. Therefore, the mean number of customers in queue is the mean number of arrivals during a mean waiting time, that is,

(21) $$L_q = \lambda W_q,$$

whence

(22) $$W_q = L_q/\lambda = \lambda/\{\mu(\mu-\lambda)\}.$$

By similar reasoning, we have

(23) $$L = \lambda W,$$

giving

(24) $$W = L/\lambda = 1/(\mu-\lambda).$$

Now, to find the distribution of the time spent in system (w_s), let a typical customer arrive to find n customers in the system. Then, for w_s to lie in $(w, w+dw)$ all the $(n+1)$ customers including the new arrival himself should complete their service within time $w+dw$. Thus, the waiting time density $f(w)$ is given by

(25) $$\begin{aligned} f(w)dw &= P\{w < w_s < w+dw\} \\ &= \sum_{n=0}^{\infty} P_n \cdot P\{n \text{ services in time } w\} \cdot P\{1 \text{ service in } dw\} \\ &= \sum_{n=0}^{\infty} (1-\rho)\rho^n \cdot \frac{(\mu w)^n e^{-\mu w}}{n!} \cdot \mu dw \end{aligned}$$

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$$= (1-\rho)\mu e^{-\mu w} dw \sum_{n=0}^{\infty} (\mu\rho w)^n/n\,!$$

$$= (1-\rho)\mu e^{-\mu w}\, dw . e^{\mu\rho w}$$

$$= (1-\rho)\mu e^{-(1-\rho)\mu w}\, dw$$

Hence, the time spent in system has an exponential distribution with mean $= 1/\{(1-\rho)\mu\} = 1/(\mu-\lambda)$, which agrees with (24).

To find the distribution of the waiting time in the queue, w_q, we see that

(26) $$P(w_q = 0) = P_0 = 1-\rho$$

and, arguing as before, the density function for the non-zero values of w_q, is given by

(27) $$f_q(w)dw = P\{w < w_q < w+dw\}$$

$$= \sum_{n=1}^{\infty} P_n . P\{(n-1) \text{ services in } w\} . P\{1 \text{ service in } dw\}$$

$$= \sum_{n=1}^{\infty} (1-\rho)\rho^n . \frac{(\mu w)^{n-1} e^{-\mu w}}{(n-1)\,!} \cdot \mu dw$$

$$= (1-\rho)\rho\mu e^{-\mu w}\, dw . e^{\mu\rho w}$$

$$= \rho\mu(1-\rho)e^{-(1-\rho)\mu w}\, dw$$

From (26), (27) we see that the distribution of w_q consists of a discrete atom at zero and a continuous portion which is a truncated exponential distribution over $(0+, \infty)$. The mean waiting time in the queue is

(28) $$W_q = 0.(1-\rho) + \int_{0+}^{\infty} w.\rho\mu(1-\rho)e^{-(1-\rho)\mu w}\, dw$$

$$= \frac{\rho}{\mu(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

(c) *The busy period*

The distribution of busy period, which involves time-dependent probabilities, will be discussed in the next chapter. However, the mean duration of a busy period can be obtained from simple arguments as follows.

The server's entire time is made up of alternating busy and idle periods; so the number of busy and idle periods in a long period of time T can be taken to be equal. Now, an idle period is the time from the instant the server

becomes idle until the next arrival which, according to the Markovian property of the exponential inter-arrival time distribution, is distributed as the entire inter-arrival time. Hence the mean duration of an idle period is $1/\lambda$. Now, the probability that the server is idle is $P_0 = 1-\rho$. Therefore, over the time T the server is expected to be idle for a duration $T(1-\rho)$, which is equivalent to $T\lambda(1-\rho)$ idle periods. Thus the server's busy time $T\rho$ is made up of $T\lambda(1-\rho)$ busy periods, and the mean duration of a busy period is

(29) $$T\rho/[T\lambda(1-\rho)] = 1/(\mu-\lambda)$$

It follows that the mean number of customers served in a busy period is

(30) $$\mu/(\mu-\lambda) = (1-\rho)^{-1}$$

(d) *Higher Moments*

In the foregoing discussion of the queue length, waiting time, etc., we have kept the analysis up to the derivation of the mean, which is the first moment, about the origin, of the distribution in question. However, it is often easy to evaluate higher moments like the second, third, fourth moments about the origin or other values, and so on, by using their definitions or by successive differentiation of the Laplace transforms or generating functions as the case may be. We give below derivations of the variances.

(i) *System size.* Let N be the r.v. denoting the number in system. Then, the variance of N is

(31) $$\text{Var}[N] = E[N^2] - (E[N])^2$$

where $E[X]$ stands for expected value of X, and $E[N] = L$, given by (18), (19). Also, by definition,

(32) $$E[N^2] = \sum_{n=0}^{\infty} n^2 P_n$$

$$= (1-\rho) \sum_{n=0}^{\infty} n^2\rho^n \quad \text{(on substituting for } P_n\text{)}$$

$$= (1-\rho)\left[\frac{\rho}{(1-\rho)^2} + \frac{2\rho^2}{(1-\rho)^3}\right]$$

$$= \frac{\rho}{1-\rho} + \frac{2\rho^2}{(1-\rho)^2}$$

Thus, (31) simplifies to

(33) $$\text{Var}[N] = \frac{\rho}{1-\rho} + \frac{\rho^2}{(1-\rho)^2}$$

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It is interesting to note that the factorial moment, of order 2, of the system size distribution is

$$E[N(N-1)] = E[N^2]-E[N] \tag{34}$$
$$= \frac{2\rho^2}{(1-\rho)^2}$$

(ii) *Queue length.* If N_q, N_{serv} denote the number in queue and number in service, respectively, we have

$$N = N_q+N_{serv} \tag{35}$$

and from the description of the system it is clear that N_q and N_{serv} are independent. Hence,

$$\mathrm{Var}[N] = \mathrm{Var}[N_q]+\mathrm{Var}[N_{serv}]. \tag{36}$$

Now, the r.v. N_{serv} has the probability distribution

$$\begin{cases} P(N_{serv}=0) = P_0 = 1-\rho \\ P(N_{serv}=1) = \rho \end{cases} \tag{37}$$

whence it is easy to show that

$$\mathrm{Var}[N_{serv}] = \rho(1-\rho) \tag{38}$$

From (35) and (38), we have

$$\mathrm{Var}[N_q] = \frac{\rho}{1-\rho}+\frac{\rho^2}{(1-\rho)^2}-\rho(1-\rho) \tag{39}$$

(iii) *Time in system.* Since w_s, the waiting time in system, has a negative exponential distribution with mean $1/(\mu-\lambda)$, (see (25)), its variance is given by

$$\mathrm{Var}[w_s] = 1/(\mu-\lambda)^2 \tag{40}$$

Incidentally, it is easy to see that the r-th moment of w_s *about the origin* is

$$\mu_r' = \int_0^\infty w^r f(w)dw \tag{41}$$
$$= r!/(\mu-\lambda)^r$$

$f(w)$ in (41) being given by (25).

(iv) *Waiting time in queue.* Proceeding as in (35), (36), we have for variance of w_q, the waiting time in queue,

$$\mathrm{Var}[w_q] = \mathrm{Var}[w_s]-\mathrm{Var}[v], \ v \text{ being the service time} \tag{42}$$
$$= \frac{1}{(\mu-\lambda)^2}-\frac{1}{\mu^2}$$

Note that (42) can also be easily derived from (27).

1.6 State-Dependent Parameters

We now consider the same queueing model as in Section 1.5 except that the mean arrival and service rates λ and μ will now be taken to depend upon the state n of the system and, therefore, will be denoted by λ_n and μ_n respectively. By choosing different forms of the functions λ_n and μ_n, we shall obtain various interesting models of queues with Poisson arrivals and exponential service times.

Proceeding as before, eq. **1.5** (4), (6), now become

$$\frac{d}{dt}P_n(t) = -(\lambda_n+\mu_n)P_n(t)+\lambda_{n-1}P_{n-1}(t)+\mu_{n+1}P_{n+1}(t), \ (n = 1, 2, \ldots) \tag{1}$$

$$\frac{d}{dt}P_0(t) = -\lambda_0P_0(t)+\mu_1P_1(t) \tag{2}$$

Hence, the steady-state equations are

$$0 = -(\lambda_n+\mu_n)P_n+\lambda_{n-1}P_{n-1}+\mu_{n+1}P_{n+1}, \ (n = 1, 2, \ldots) \tag{3}$$

$$0 = -\lambda_0P_0+\mu_1P_1 \tag{4}$$

From (4), we have

$$P_1 = \frac{\lambda_0}{\mu_1}P_0 \tag{5}$$

Writing eq. (3) for $n = 1$, substituting for P_1 from (5) and solving for P_2, we have

$$P_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}P_0 \tag{6}$$

and so on.

By mathematical induction, we can prove that

$$P_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}P_0 \tag{7}$$

In the general case when there is no upper limit on n, we have the normalizing equation

$$P_0\left[1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\ldots+\frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}+\ldots\right] = 1 \tag{8}$$

For a steady-state solution to be possible, the infinite series inside the brackets in (8) should be convergent. Let it have a sum S; then

$$P_0 = 1/S \tag{9}$$

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From (7) and (9) we have the steady-state solution

$$P_n = \frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n} S^{-1}, \quad n = 0, 1, 2, \ldots \tag{10}$$

where an empty product is taken as unity.

We proceed to consider a few particular cases of the model.

(a) *The queueing system* $M/M/1/N$

To obtain the solution for the queueing system with mean arrival and service rates λ and μ, respectively, where no more than N units are allowed to enter the system at any time, we set

$$\begin{aligned} \lambda_n &= \lambda \ (0 \leqslant n < N) \\ &= 0 \ (n \geqslant N) \\ \mu_n &= \mu \ (n = 1, 2, \ldots) \end{aligned} \tag{11}$$

Then, from (8), we see that S is the finite sum

$$S = 1+\rho+\rho^2+\ldots+\rho^N = \frac{1-\rho^{N+1}}{1-\rho} \tag{12}$$

where $\rho = \lambda/\mu$, the traffic intensity. Since S is finite, we conclude that the steady state exists in all cases, *i.e.*, for all values of ρ. Substituting for S in (10), we have

$$P_n = (1-\rho)\rho^n/(1-\rho^{N+1}), \ \rho \neq 1, \ n = 0, 1, 2, \ldots, N \tag{13}$$

When $\rho = 1$, (12) gives $S = N+1$ and, therefore,

$$P_n = 1/(N+1), \ \rho = 1, \ n = 0, 1, \ldots, N \tag{14}$$

which can also be obtained from (13) by letting $\rho \to 1$.

The mean number in the system is

$$\begin{aligned} L &= \sum_{n=0}^{N} nP_n \\ &= \rho[1-(N+1)\rho^N+N\rho^{N+1}]/[(1-\rho)(1-\rho^{N+1})], \ (\rho \neq 1) \\ &= N/2, \ (\rho = 1) \end{aligned} \tag{15}$$

The mean number in the queue is

$$\begin{aligned} L_q &= \sum_{n=1}^{N} (n-1)P_n \\ &= \rho^2[1-N\rho^{N-1}+(N-1)\rho^N]/[(1-\rho)(1-\rho^{N+1})], \ (\rho \neq 1) \\ &= N(N-1)/\{2(N+1)\}, \ (\rho = 1) \end{aligned} \tag{16}$$

(b) *The queueing system* $M/M/c/\infty$

Let there be c severs (channels), the service times at each of which are exponentially distributed with mean service rate μ and let arrivals be Poisson with mean rate λ. To obtain the solution for this case, we set

$$\begin{aligned} \lambda_n &= \lambda \quad \text{for all } n \\ \mu_n &= n\mu \quad \text{for } n < c \\ &= c\mu \quad \text{for } n \geqslant c \end{aligned} \tag{17}$$

The justification for the above substitutions is that when n servers are busy, the probability of a departure during $(t, t+\Delta t)$ is the probability that any one of the n servers completes service which, by the addition theorem of probability, is $n\mu\Delta t$.

In this case, we have, from (8),

$$S = 1+\frac{\lambda}{\mu}+\frac{\lambda^2}{2!\,\mu^2}+\ldots+\frac{\lambda^{c-1}}{(c-1)!\,\mu^{c-1}}+\frac{\lambda^c}{c!\,\mu^c}\left\{1+\frac{\lambda}{c\mu}+\frac{\lambda^2}{(c\mu)^2}+\ldots\right\} \tag{18}$$

For a c-channel system it is customary to write the traffic intensity

$$\rho = \lambda/(c\mu) \tag{19}$$

Thus (18) gives

$$\begin{aligned} S &= 1+c\rho+\frac{(c\rho)^2}{2!}+\ldots+\frac{(c\rho)^{c-1}}{(c-1)!}+\frac{(c\rho)^c}{c!}\{1+\rho+\rho^2+\ldots\} \\ &= \sum_{j=0}^{c-1} (c\rho)^j/j!+\{(c\rho)^c/c!\}(1-\rho)^{-1}, \text{ provided } \rho < 1 \end{aligned} \tag{20}$$

The series in (20) is convergent only when $\rho < 1$, *i.e.*, when $\lambda/c\mu < 1$. We conclude that the steady state exists only when $\rho < 1$.

Using (17) in (7), we have

$$\begin{aligned} P_n &= P_0 \cdot c^n\rho^n/n! \text{ for } n < c \\ &= P_0 \cdot c^c\rho^n/c! \text{ for } n \geqslant c \end{aligned} \tag{21}$$

where

$$P_0 = S^{-1} = \left[\sum_{j=0}^{c-1} \{(c\rho)^j/j!\}+\{(c\rho)^c/(c!\,(1-\rho))\}\right]^{-1}. \tag{22}$$

It is easy to see that the probability of having to wait is

$$\begin{aligned} P(n \geqslant c) &= P_0(c^c/c!) \sum_{n=c}^{\infty} \rho^n \\ &= P_0(c\rho)^c/\{c!\,(1-\rho)\} = P_c/(1-\rho) \end{aligned} \tag{23}$$

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The expected number waiting in the queue is

$$L_q = \sum_{n=c}^{\infty} (n-c)P_n \tag{24}$$

$$= P_0[(c\rho)^c/c\,!] \sum_{n=c}^{\infty} (n-c)\rho^{n-c}$$

$$= P_0\rho(c\rho)^c/\{c\,!\,(1-\rho)^2\}$$

$$= \frac{\lambda\mu(\lambda/\mu)^c P_0}{(c-1)\,!\,(c\mu-\lambda)^2}$$

The mean number of customers being served or the mean number of busy channels is

$$L_s = \sum_{n=0}^{c-1} nP_n + \sum_{n=c}^{\infty} cP_n \tag{25}$$

$$= \sum_{n=1}^{c} nP_n + \sum_{n=c+1}^{\infty} cP_n$$

$$= \sum_{n=1}^{c} n.P_0 \frac{(c\rho)^n}{n\,!} + \sum_{n=c+1}^{\infty} c.P_0 \frac{c^c\rho^n}{c\,!}$$

$$= c\rho\, P_0 \left[\sum_{n=1}^{c} \frac{(c\rho)^{n-1}}{(n-1)\,!} + \frac{c^c}{c\,!} \cdot \frac{\rho^c}{1-\rho} \right]$$

$$= c\rho\, P_0 \left[\sum_{j=0}^{c-1} \frac{(c\rho)^j}{j\,!} + \frac{(c\rho)^c}{c\,!\,(1-\rho)} \right]$$

$$= c\rho\, P_0 P_0^{-1}$$

$$= c\rho = \lambda/\mu$$

Hence, the mean number in the system is

$$L = L_q + L_s = L_q + (\lambda/\mu) \tag{26}$$

The mean waiting time in queue is

$$W_q = L_q/\lambda = \mu(\lambda/\mu)^c P_0/\{(c-1)\,!\,(c\mu-\lambda)^2\} \tag{27}$$

The mean time spent in the system is

$$W = L/\lambda = W_q + (1/\mu) \tag{28}$$

(c) *The system $M/M/\infty$*

This is the system where every arrival is able to find a server and is sometimes called the ample-server system. Possible applications are self-service systems where the customer carries the server with himself. In such systems

no queueing is involved and some authors like Conolly (1975) have pointed out that they should not be called queueing systems. Here we have

$$\lambda_n = \lambda \quad \text{for all } n \tag{29}$$

$$\mu_n = n\mu, \quad n = 0, 1, 2, \ldots$$

Therefore,

$$S = 1 + (\lambda/\mu) + \frac{(\lambda/\mu)^2}{2\,!} + \ldots = e^{\lambda/\mu} \tag{30}$$

Note that the series is convergent and, therefore, the steady state exists for all values of λ/μ. It is easily seen that we have

$$P_n = \frac{(\lambda/\mu)^n e^{-\lambda/\mu}}{n\,!} \tag{31}$$

so that the number in the system has a Poisson distribution with mean (λ/μ). Thus

$$L = \lambda/\mu \tag{32}$$

$$W = L/\lambda = 1/\mu \tag{33}$$

which is otherwise obvious : since there is no queueing, the mean time in system equals the mean service time.

(d) *Queue with balking*

Let us consider the M/M/1 queueing system with mean arrival and service rates λ and μ, such that when there are n units in the system, an incoming customer joins the queue with probability $1/(1+n)$. So we take

$$\lambda_n = \lambda/(n+1), \quad n = 0, 1, 2, \ldots \tag{34}$$

$$\mu_n = \mu \quad \text{for all } n$$

Then,

$$S = 1 + (\lambda/\mu) + \frac{(\lambda/\mu)^2}{2\,!} + \ldots = e^{\lambda/\mu} \tag{35}$$

as before. Hence, in this case also, the number in the system has the Poisson distribution, given by (31). It follows that the mean number in the system is λ/μ, as before. But in the present case queueing is allowed, and we have the mean number in the queue

$$L_q = \sum_{n=1}^{\infty} (n-1)P_n = (\lambda/\mu) + e^{-\lambda/\mu} - 1 \tag{36}$$

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(e) *The system M/M/c/c (loss system)*

This is the system with c channels, where no queueing is allowed, so that the maximum number allowed in the system is c. An application of this model is a telephone exchange with c trunk lines with no facility for holding subscribers who fail to get a free line. Taking λ as the mean arrival rate and μ as the mean service rate for each channel, we have

$$\lambda_n = \lambda \quad \text{for } n < c \tag{37}$$
$$= 0 \quad \text{for } n \geqslant c$$
$$\mu_n = n\mu \quad \text{for } n \leqslant c$$

Therefore, S is the finite sum

$$S = 1+(\lambda/\mu)+\frac{(\lambda/\mu)^2}{2\,!}+\ldots+\frac{(\lambda/\mu)^c}{c\,!} \tag{38}$$

Thus, the steady state exists for all values of λ/μ, and we have

$$P_n = \frac{(\lambda/\mu)^n}{n\,!}\,S^{-1}, \quad (0 \leqslant n \leqslant c) \tag{39}$$

where S is given by (38). The probability that all the channels are busy and, therefore, the incoming calls are lost is given by

$$P_c = \frac{\dfrac{(\lambda/\mu)^c}{c\,!}}{1+(\lambda/\mu)+\dfrac{(\lambda/\mu)^2}{2\,!}+\ldots+\dfrac{(\lambda/\mu)^c}{c\,!}} \tag{40}$$

(40) is called *Erlang's loss formula* and is still in use in telephone traffic work.

(f) *The machine interference model*

Let the customers come from a finite source of k members, such that the probability that any member will call for service, independent of other members, in the interval $(t, t+\Delta t)$ is $\lambda\Delta t+o(\Delta t)$, and let there be a single server with mean service rate μ, the service times being exponentially distributed. Then, we have

$$\lambda_n = (k-n)\lambda \quad \text{for } n \leqslant k \tag{41}$$
$$= 0 \quad \text{for } n > k$$
$$\mu_n = \mu \quad \text{for } n = 1, 2, \ldots$$

Therefore,

$$S = 1+k(\lambda/\mu)+k(k-1)(\lambda/\mu)^2+\ldots+k\,!\,(\lambda/\mu)^k \tag{42}$$

and

$$P_0 = 1/S \tag{43}$$

$$P_n = k(k-1)\ldots(k-n+1)(\lambda/\mu)^n P_0, \ (n = 1, 2, \ldots, k) \tag{44}$$

Possible applications of this model are situations where there are k automatic machines under the charge of a single operator and the machines fail and require the operator's attention from time to time, or situations where k industrial workers require service at a facility like a grinding machine, tool crib, etc.

(g) *The brand-share model*

Consider a market consisting of N consumers, out of which at any time n consumers are using a particular brand, say brand A, of a product. Let the consumers switch into and out of brand A in a Poisson fashion with rates λ and μ, respectively, so that when the brand-share of A is n, we can write

$$\lambda_n = (N-n)\lambda \quad \text{for } n \leqslant N \tag{45}$$
$$= 0 \quad \text{for } n > N$$
$$\mu_n = n\mu \quad \text{for } n \leqslant N$$

Thus, we have

$$S = 1+\binom{N}{1}(\lambda/\mu)+\binom{N}{2}(\lambda/\mu)^2+\ldots+\binom{N}{N}(\lambda/\mu)^N \tag{46}$$
$$= \{1+(\lambda/\mu)\}^N$$

whence

$$P_0 = 1/S = \{\mu/(\lambda+\mu)\}^N \tag{47}$$

$$P_n = \binom{N}{n}(\lambda/\mu)^n P_0 \tag{48}$$

Clearly P_n is the probability that A has a brand share n, while

$$P_N = \{\lambda/(\lambda+\mu)\}^N \tag{49}$$

is the probability of market saturation by brand A. It may be noted that P_n follows the binomial probability distribution :

$$P_n = \binom{N}{n} q^{N-n} p^n \tag{50}$$

where

$$p = \lambda/(\lambda+\mu), \ q = 1-p \tag{51}$$

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Hence, the expected brand share of A is

$$E(n) = Np = N\lambda/(\lambda+\mu) \tag{52}$$

Ex. 1. A television repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrivals of sets are approximately Poisson with an average rate of 10 per 8-hour day, what is the repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ? [By permission from Sasieni, Yaspan and Friedman (1959)]

We have an M/M/1 queueing system with the mean arrival rate

$$\lambda = \frac{10}{8} = \frac{5}{4} \text{ sets per hour}$$

and the mean service rate

$$\mu = 2 \text{ sets per hour}$$

Therefore,

$$\rho = \frac{\lambda}{\mu} = \frac{5}{8}$$

The probability that the server is idle

$$= P_0 = 1-\rho = \frac{3}{8}$$

Hence the repairman's (server's) expected idle time per day

$$= \frac{3}{8} \times \text{duration of a day (}i.e.\text{, working hours)}$$

$$= \frac{3}{8} \times 8 \text{ hours} = 3 \text{ hours}$$

Also, expected number of units in the system

$$= L = \frac{\rho}{1-\rho} = \frac{5/8}{1-5/8} = \frac{5}{3} = 1\frac{2}{3} \text{ jobs (sets)}$$

Ex. 2. A bank has two tellers working on savings accounts. The first teller handles withdrawals only and the second teller handles deposits only. It has been observed that the service time distributions for both deposits and withdrawals are exponential with mean service time 3 minutes per customer. Depositors are found to arrive in a Poisson fashion throughout the day with mean arrival rate 16 per hour. Withdrawers also arrive in a Poisson fashion with mean arrival rate 14 per hour. What would be the

effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits ? What would be the effect if this could only be accomplished by increasing the mean service time to 3·5 minutes ? [By permission from Sasieni, Yaspan and Friedman (1959)]

Originally we have for depositors, $\lambda = 16$ arrivals per hour, $\mu = 20$ services per hour, so that $\rho = \lambda/\mu = 4/5 < 1$ (steady state exists).

Therefore, $$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{16}{20(20-16)} = \frac{1}{5} \text{ hour} = 12 \text{ minutes}$$

And, for withdrawers, $\lambda = 14$ per hour, $\mu = 20$ per hour, $\rho = 7/10 < 1$

Therefore, $$W_q = \frac{14}{20\times 6} \text{ hour} = 7 \text{ minutes}$$

In the second case, we have an M/M/2 system with $\lambda = 16+14 = 30$ per hour, $\mu = 20$ per hour for each server, so that

$$\rho = \lambda/2\mu = \frac{3}{4} < 1 \text{ (so steady state exists)}$$

Therefore, from 1.6 (22), taking $c = 2$, we have

$$P_0 = \left[\sum_{j=0}^{1} \frac{(3/2)^j}{j\,!} + \left(\frac{3}{2}\right)^2 \Big/ \left\{2\,!\left(1-\frac{3}{4}\right)\right\}\right]^{-1}$$

$$= \left[1+\frac{3}{2}+\frac{9}{2}\right]^{-1} = \frac{1}{7}$$

Hence, from 1.6(27),

$$W_q = \frac{\mu(\lambda/\mu)^2}{(2\mu-\lambda)^2} P_0 = \frac{20\times\frac{9}{4}}{100} \times \frac{1}{7} \text{ hour} = 3\frac{6}{7} \text{ min.}$$

In the third case, when the mean service time increases to 3·5 min., we have $\mu = \frac{120}{7}$ services per hour, and $\lambda = 30$ per hour, as before. Therefore, $\rho = \lambda/2\mu = 7/8 < 1$. Thus, we have

$$P_0 = \left[1+\frac{7}{4}+\frac{(7/4)^2}{2\times\frac{1}{8}}\right]^{-1} = \left[1+\frac{7}{4}+\frac{49}{4}\right]^{-1} = 1/15$$

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whence

$$W_q = \frac{\frac{120}{7}\,(7/4)^2}{\left(\frac{240}{7}-30\right)^2} \times \frac{1}{15} \text{ hour} = \frac{343}{30} \text{ min.} = 11\frac{13}{30} \text{ min.}$$

1.7 Alternative Solution Methods

We give below two alternative methods for the solution of the set of equations 1.5(7)–(8) for the steady-state queue length probabilities P_n in M/M/1.

(a) *Difference-equation method*

The equations can be written

$$\mu P_{n+2} - (\lambda+\mu)P_{n+1} + \lambda P_n = 0, \quad n = 0, 1, 2, \ldots \tag{1}$$

$$\mu P_1 - \lambda P_0 = 0 \tag{2}$$

Now, (1) is a difference equation with constant coefficients. Using the difference operator E defined by

$$E^r P_n = P_{n+r}, \quad r = 1, 2, \ldots$$

we can write (1) in the form

$$[\mu E^2 - (\lambda+\mu)E + \lambda]P_n = 0$$

so that the characteristic equation is

$$\mu E^2 - (\lambda+\mu)E + \lambda = 0$$

or,

$$(\mu E - \lambda)(E-1) = 0$$

which gives

$$E = 1, \rho, \text{ where } \rho = \lambda/\mu$$

Hence, the solution of (1) is

$$P_n = A + B\rho^n \tag{3}$$

where A, B are arbitrary constants.

Now, substituting from (3) in (2), we have

$$A + B\rho = \rho(A+B)$$

which gives $A = 0$ and hence $B = P_0$. Thus we have, from (3),

$$P_n = P_0\rho^n \tag{4}$$

which is the same as 1.5(13), and we can proceed further as before to show that $P_0 = 1-\rho$ $(\rho < 1)$, which completes the solution.

(b) *Generating-function method*

Let us define the generating function

$$\mathrm{P}(z) = \sum_{n=0}^{\infty} P_n z^n, \quad |z| \leqslant 1 \tag{5}$$

Multiplying eq. 1.5(7) by z^n, summing over $n = 1, 2, \ldots$, and adding 1.5(8), we have

$$0 = -(\lambda+\mu)\sum_{n=0}^{\infty} P_n z^n + \mu P_0 + \lambda \sum_{n=1}^{\infty} P_{n-1} z^n + \mu \sum_{n=0}^{\infty} P_{n+1} z^n \tag{6}$$

or,

$$0 = -(\lambda+\mu)P(z) + \mu P_0 + \lambda z P(z) + (\mu/z)\{P(z) - P_0\} \tag{7}$$

or,

$$\{\lambda z^2 - (\lambda+\mu)z + \mu\}P(z) + \mu(z-1)P_0 = 0 \tag{8}$$

This yields

$$\mathrm{P}(z) = \frac{\mu(1-z)P_0}{\lambda z^2 - (\lambda+\mu)z + \mu} \tag{9}$$

The denominator of the right-hand member of (9) can be factorized as

$$\lambda z^2 - (\lambda+\mu)z + \mu = (\lambda z - \mu)(z-1)$$

so that (9) can be written

$$P(z) = \frac{\mu P_0}{\mu - \lambda z} = P_0(1-\rho z)^{-1} = P_0 \sum_{n=0}^{\infty} (\rho z)^n, \text{ since } \rho z < 1 \tag{10}$$

Recalling the definition (5) and equating the coefficients of z^n on the two sides, we have

$$P_n = P_0\rho^n \tag{11}$$

P_0 can be evaluated as in 1.5. Alternatively, we may note that $P(1) = 1$, and then (10) gives $P_0 = 1-\rho$, as before.

Ex. 3. Find the steady-state probability P_n for the finite-waiting-space multichannel queueing system M/M/c/N, where the number of channels is c and the maximum number allowed in the system is N $(N > c)$. Also deduce expressions for L, L_q, W, W_q and the expected number of busy channels.

In the notation of Section 1.6, we have

$$\begin{aligned} \lambda_n &= \lambda \quad \text{when } 0 \leqslant n < N \\ &= 0 \quad \text{when } n \geqslant N \\ \mu_n &= n\mu \quad \text{when } n < c \\ &= c\mu \quad \text{when } n \geqslant c \end{aligned}$$

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Therefore, writing $\lambda/c\mu = \rho$, we have as in 1.6(21)

$$P_n = P_0 c^n \rho^n / n\,!, \quad (0 \leqslant n < c)$$
$$= P_0 c^c \rho^n / c\,!, \quad (c \leqslant n \leqslant N)$$

where (as in 1.6(18))

$$S = P_0^{-1} = \sum_{j=0}^{c-1} \frac{(c\rho)^j}{j\,!} + \frac{(c\rho)^c}{c\,!} \sum_{i=0}^{N-c} \rho^i$$

$$= \sum_{j=0}^{c-1} \frac{(c\rho)^j}{j\,!} + \frac{(c\rho)^c}{c\,!} \cdot \frac{1-\rho^{N-c+1}}{1-\rho} \quad \text{when } \rho \neq 1$$

$$= \sum_{j=0}^{c-1} \frac{c^j}{j\,!} + \frac{c^c}{c\,!}(N-c+1) \quad \text{when } \rho = 1$$

Note that the steady state exists for all values of ρ.

Now, the mean number in the queue (waiting) is

$$L_q = \sum_{n=c+1}^{N} (n-c)P_n = \{P_0(c\rho)^c/c\,!\} \sum_{n=c+1}^{N} (n-c)\rho^{n-c}$$

$$= \{P_0(c\rho)^c/c\,!\}\{\rho + 2\rho^2 + 3\rho^3 + \ldots + (N-c)\rho^{N-c}\}$$

$$= \frac{P_0(c\rho)^c\rho}{c\,!\,(1-\rho)^2}[1-\rho^{N-c} - (1-\rho)(N-c)\rho^{N-c}]$$

The average number being served or the mean number of busy channels is

$$L_s = \sum_{n=0}^{c-1} nP_n + \sum_{n=c}^{N} cP_n$$

$$= \sum_{n=1}^{c} nP_0 \cdot \frac{(c\rho)^n}{n\,!} + \sum_{n=c+1}^{N} cP_0 \frac{c^c\rho^n}{c\,!}$$

$$= c\rho P_0 \left[\sum_{n=1}^{c} \{(c\rho)^{n-1}/(n-1)\,!\} + \frac{c^c}{c\,!} \sum_{n-c+1}^{N} \rho^{n-1} \right]$$

$$= c\rho P_0 \left[\sum_{j=0}^{c-1} (c\rho)^j/j\,! + \frac{(c\rho)^c}{c\,!} \sum_{n=0}^{N-c-1} \rho^n \right]$$

$$= c\rho P_0 \left[P_0^{-1} - \frac{(c\rho)^c}{c\,!} \cdot \rho^{N-c} \right] = c\rho(1-P_N) = (\lambda/\mu)(1-P_N)$$

Therefore, the mean number in the system is

$$L = L_q + L_s = L_q + (\lambda/\mu)(1-P_N)$$

The effective arrival rate at the system is $\lambda' = \lambda(1-P_N)$. Therefore, from Little's formula,

$$W = L/\lambda' = W_q + (1/\mu)$$
$$W_q = L_q/\lambda'$$

Ex. 4. At a port there are six unloading berths and four unloading crews. When all the berths are full, arriving ships are diverted to an overflow facility 20 miles down the river. Tankers arrive according to a Poisson process with a mean of one every 2 hours. It takes an unloading crew, on the average, ten hours to unload a tanker, the unloading time following an exponential distribution.

(a) On the average, how many tankers are at the port ?

(b) On the average, how long does a tanker spend at the port ?

(c) What is the average arrival rate at the overflow facility ?

Here we have an M/M/c/N system with $c = 4$ and $N = 6$. Also, $\lambda = \frac{1}{2}$ arrival per hour, and $\mu = \frac{1}{10}$ service per hour. Thus, $c\rho = \lambda/\mu = 5$, and $\rho = 5/4$. Therefore, as in Ex. 3,

$$1/P_0 = \sum_{j=0}^{3} \frac{5^j}{j\,!} + \frac{5^4}{4\,!} \cdot \frac{1-(5/4)^3}{1-(5/4)} = 1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24} \times \frac{61}{16}$$

$$\approx 138{\cdot}62$$

Therefore,

$$P_0 = 1/138{\cdot}62 \approx 0{\cdot}0072$$

$$P_N = P_0 c^c \rho^N / c\,! = 0{\cdot}0072 \times 256 \times \left(\frac{5}{4}\right)^6 / 24 = 0{\cdot}293$$

Hence,

(a) The mean number of tankers at the port is

$$L = L_q + (\lambda/\mu)(1-P_N)$$

Now,

$$L_q = \frac{P_0(c\rho)^c\rho}{c\,!\,(1-\rho)^2}[1-\rho^{N-c} - (1-\rho)(N-c)\rho^{N-c}]$$

$$= \frac{0{\cdot}0072 \times 5^4 \times (5/4)}{24 \times (1/16)}\left[1 - \frac{25}{16} + \frac{1}{4} \times 2 \times \frac{25}{16}\right] = 0{\cdot}820$$

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Thus, $L = 0{\cdot}820 + 5(1-0{\cdot}293) = 4{\cdot}36$ tankers approx.

(b) The average time spent at the port is

$$W = L/\{\lambda(1-P_N)\} = 4{\cdot}36/(\tfrac{1}{2}\times 0{\cdot}707) = 12{\cdot}33 \text{ hours}$$

Thus the average time spent in waiting before unloading is

$$W_q = W - 10 \text{ hours} = 2{\cdot}33 \text{ hours}$$

(c) The average arrival rate at the overflow facility is

$$\lambda P_N = \tfrac{1}{2}\times 0{\cdot}293 \approx 0{\cdot}15 \text{ tanker per hour}$$

PROBLEM SET 1

1. Arrivals at a telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait ?

(b) The telephone department will install a second booth when convinced that an arrival would expect to wait for at least 3 minutes for the phone. By how much should the flow of arrivals increase in order to justify a second booth ? [By permission from Sasieni, Yaspan and Friedman (1959)]

[(a) 0·3; (b) *new arrival rate one every six minutes*]

2. A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes, and if people arrive in a Poisson fashion at the rate of 10 an hour,

(a) What is the probability of having to wait for service ?

(b) What is the expected percentage of idle time for each girl ?

(c) Find the average queue length and the average number of units in the system. [By permission from Sasieni, Yaspan and Friedman (1959)]

[(a) 1/6; (b) $66\frac{2}{3}\%$; (c) 1/12; 2/3]

3. In a railway marshalling yard, goods trains arrive at the rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service-time distribution is also exponential with an average of 36 minutes, calculate the following :

(i) the mean queue size (line length).

(ii) the probability that the queue size exceeds 10.

[$\rho/(1-\rho) = 3$ *trains*; $\rho^{10} = 0{\cdot}056$]

4. At what average rate must a clerk at a supermarket work in order to ensure a probability of 0·90 that the customer will not have to wait longer than 12 minutes (a) before service, (b) including service ? Assume that there is only one counter, at which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution. [By permission from Sasieni, Yaspan and Friedman (1959)]

[(a) 2·48 *minutes per service*; (b) 2·26 *minutes per service*]

5. Problems arrive at a computer centre in a Poisson fashion at an average rate of five per day. The rules of the computer centre are that any man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of 1/3 day, and if the average solving time is inversely proportional to the number of people working on the problem, approximate the expected time in the centre for a person entering the line. [By permission from Sasieni, Yaspan and Friedman (1959)]

[8 *hours*]

6. A telephone company is planning to install telephone booths in a new airport. It has established the policy that a person should not have to wait more than 10 per cent of the times he tries to use a phone. The demand for use is estimated to be Poisson with an average of 30 per hour. The average phone call has an exponential distribution with a mean time of 5 minutes. How many phone booths should be installed ? [6]

7. Show that for the k-Erlang distribution given by 1·4(18), the mean, variance and mode are, respectively, $1/\lambda$, $1/k\lambda^2$, and $(1-1/k)(1/\lambda)$.

8. For an M/M/2 queueing system in steady state, show that

$$\lambda P_0 = \mu P_1$$
$$(\lambda+\mu)P_1 = \lambda P_0 + 2\mu P_2$$
$$(\lambda+2\mu)P_n = \lambda P_{n-1} + 2\mu P_{n+1} \quad (n = 2, 3, \ldots)$$

Hence prove that $P_n = 2\rho^n P_0$ $(n = 1, 2, \ldots)$, where $\rho = \lambda/2\mu$, and $P_0 = (1-\rho)/(1+\rho)$, and deduce that $L = 2\rho/(1-\rho^2)$, $L_q = 2\rho^2/(1-\rho^2)$, and that the probability of having to wait is $2\rho^2/(1+\rho)$.

9. Find the steady-state probability P_n, given that

$$\lambda_n = \begin{cases} (N-n)\lambda, & 0 \leqslant n < N \\ 0, & n \geqslant N \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 1 \leqslant n < c \\ c\mu, & n \geqslant c \end{cases}$$

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Describe a practical situation represented by the above model.

$$\left[P_n = \binom{N}{n} (\lambda/\mu)^n P_0, \ 0 \leqslant n < c \right.$$

$$\left. = \binom{N}{n} (n!/c!\, c^{n-c})(\lambda/\mu)^n P_0, \ c \leqslant n \leqslant N \right]$$

10. In Problem 3, assume that the yard can admit 9 trains at a time (there being 10 lines, one of which is reserved for shunting purposes). Calculate the probability that the yard is (i) empty, (ii) completely occupied, and find the mean number of trains in the yard. 1867 78

[0·26; 0·02; 2·4 *trains*]

11. For an M/M/1 system, show by simple arguments that $W_q = L/\mu$, and $W = (L+1)/\mu$. Hence use the expression for L to deduce those for W_q and W.

12. At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while another is given signal to leave the station. Trains arrive at the station at an average of 6 per hour, and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady-state probabilities for the various number of trains in the system. Also find the average waiting time of a new train coming into the yard.

[$P_0 = 0{\cdot}533$, $W_q = 3\frac{2}{3}$ *minutes*]

CHAPTER 2

POISSON QUEUES II

2.1 Introduction

In Chapter 1, we obtain the steady-state solutions of various queueing systems with Poisson arrivals and exponential service times like M/M/1, M/M/c, etc. In this chapter we shall discuss the time-dependent or transient solutions of similar queueing systems.

2.2 The Queueing System M/M/1—Transient Solution

As we saw in Section 1.5, the probabilities $P_n(t)$ that there are n units in the system at time t are given by (Cf. 1.5(4), (6))

$$\frac{d}{dt} P_n(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \ n = 1, 2, \ldots \tag{1}$$

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t) + \mu P_1(t) \tag{2}$$

Define the p.g.f. (probability generating function)

$$G(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n \tag{3}$$

which clearly converges inside and on the unit circle $|z| = 1$. Multiplying (1) by z^n, summing over n and adding (2), we have, on simplification,

$$\frac{d}{dt} G(z, t) = -(\lambda+\mu-\lambda z-\mu/z)G(z, t) + \mu(1-1/z)P_0(t) \tag{4}$$

Let the system start at time $t = 0$ with i units in the system, so that

$$P_n(0) = \delta_{in} = \begin{cases} 1, & \text{when } n = i \\ 0, & \text{when } n \neq i \end{cases} \tag{5}$$

δ_{in} is called the Kronecker delta. Thus, we have

$$G(z, 0) = z^i \tag{6}$$

Let $g(z, s)$ be the Laplace transform (L.T.) of $G(z, t)$, defined by

$$g(z, s) = \int_0^{\infty} e^{-st} G(z, t)\, dt, \ \mathrm{Re}(s) > 0 \tag{7}$$

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and let the L.T.'s of other functions be similarly denoted by the corresponding lower case letters. From (4), on taking L.T.'s, we have

$$(8) \qquad (s+\lambda+\mu-\lambda z-\mu/z)g(z,s) = z^{i}+\mu(1-1/z)p_0(s)$$

or,

$$(9) \qquad g(z,s) = \frac{z^{i+1}-\mu(1-z)p_0(s)}{(s+\lambda+\mu)z-\lambda z^2-\mu}$$

Since $g(z, s)$ converges inside and on the unit circle for $\mathrm{Re}(s) > 0$, the zeros of the denominator inside and on $|z| = 1$ must coincide with the corresponding zeros of the numerator. The zeros of the denominator are

$$(10) \qquad \alpha_1(s) = (2\lambda)^{-1}[s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}]$$

$$(11) \qquad \alpha_2(s) = (2\lambda)^{-1}[s+\lambda+\mu+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}]$$

Now, let

$$(12) \qquad f(z) = (s+\lambda+\mu)z$$
$$g(z) = -(\lambda z^2+\mu)$$

Then, on the unit circle $|z| = 1$,

$$(13) \qquad |f(z)| = |s+\lambda+\mu| > |\lambda+\mu| = g(z)$$

Hence, by Rouché's theorem, $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the unit circle. But $f(z)$ has only one zero, and so $f(z)+g(z)$ also has only one zero inside the unit circle. This zero must be α_1. Thus we have

$$(14) \qquad p_0(s) = \frac{\alpha_1^{i+1}}{\mu(1-\alpha_1)}$$

Therefore,

$$(15) \qquad g(z,s) = \frac{z^{i+1}-(1-z)\cdot\dfrac{\alpha_1^{i+1}}{1-\alpha_1}}{-\lambda(z-\alpha_1)(z-\alpha_2)}$$

or,

$$g(z,s) = \frac{z^{i+1}(1-\alpha_1)-(1-z)\alpha_1^{i+1}}{\lambda\alpha_2(z-\alpha_1)(1-z/\alpha_2)(1-\alpha_1)}$$

$$= \frac{(z^{i+1}-\alpha_1^{i+1})-z\alpha_1(z^{i}-\alpha_1^{i})}{\lambda\alpha_2(z-\alpha_1)(1-z/\alpha_2)(1-\alpha_1)}$$

$$= \frac{(z^{i}+\alpha_1 z^{i-1}+\ldots+\alpha_1^{i})-z\alpha_1(z^{i-1}+\alpha_1 z^{i-2}+\ldots+\alpha_1^{i-1})}{\lambda\alpha_2(1-z/\alpha_2)(1-\alpha_1)}$$

$$= \frac{(z^{i}+\alpha_1 z^{i-1}+\ldots+\alpha_1^{i})(1-\alpha_1)+\alpha_1^{i+1}}{\lambda\alpha_2(1-z/\alpha_2)(1-\alpha_1)}$$

or, using binomial expansion,

$$(16) \qquad g(z,s) = \frac{1}{\lambda\alpha_2}(z^{i}+\alpha_1 z^{i-1}+\ldots+\alpha_1^{i})\sum_{k=0}^{\infty}(z/\alpha_2)^k + \frac{\alpha_1^{i+1}}{\lambda\alpha_2(1-\alpha_1)}\sum_{k=0}^{\infty}(z/\alpha_2)^k$$

Note that $|z/\alpha_2| < 1$.

No, $p_n(s)$, the L.T. of $P_n(t)$ is the coefficient of z^n in $g(z, s)$. The contribution to the coefficient of z^n from the second term on the right-hand side of (16) is

$$(17) \qquad \frac{\alpha_1^{i+1}}{\lambda\alpha_2^{n+1}(1-\alpha_1)} = \frac{\alpha_1^{i+1}}{\lambda\alpha_2^{n+1}}(1+\alpha_1+\alpha_1^2+\ldots)$$

$$= \frac{1}{\lambda}\cdot\frac{\alpha_1^{n+i+2}}{(\alpha_1\alpha_2)^{n+1}}\sum_{k=0}^{\infty}\alpha_1^k$$

$$= \frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\alpha_1^k$$

$$= \frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\left(\frac{\mu}{\lambda}\right)^k\frac{1}{\alpha_2^k}, \quad \text{since } \alpha_1\alpha_2 = \mu/\lambda$$

And, the contribution from the first term is

$$(18) \qquad \sum_{m=(i-n)^+}^{i}\frac{1}{\lambda\alpha_2}\cdot\frac{\alpha_1^m}{\alpha_2^{n-i+m}} = \sum_{m=(i-n)^+}^{i}\frac{(\mu/\lambda)^m}{\lambda\alpha_2^{n-i+2m+1}}$$

where

$$(19) \qquad (x)^+ = \max(x, 0)$$

Therefore,

$$(20) \qquad p_n(s) = \frac{1}{\lambda}\left[\sum_{m=(i-n)^+}^{i}\frac{(\mu/\lambda)^m}{\alpha_2^{n-i+2m+1}} + \left(\frac{\lambda}{\mu}\right)^{n+1}\sum_{k=n+i+2}^{\infty}\frac{(\mu/\lambda)^k}{\alpha_2^k}\right]$$

Now, from Appendix E, eq. (18), we have, for the inverse Laplace transform,

$$(21) \qquad \mathscr{L}^{-1}[(s+\sqrt{s^2-a^2})^{-\nu}] = \nu a^{-\nu}\, t^{-1} I_\nu(at)$$

where $I_\nu(z)$ is the modified Bessel function of the first kind and order ν, given by

$$(22) \qquad I_\nu(z) = \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}$$

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Therefore,

$$\mathcal{L}^{-1}\frac{1}{\alpha_k^k} = e^{-(\lambda+\mu)t}\cdot(2\lambda)^k\cdot k(2\sqrt{\lambda\mu}\,t)^{-k}I_k(2\sqrt{\lambda\mu}\,t) \tag{23}$$

$$= e^{-(\lambda+\mu)t}\cdot(\sqrt{\lambda}/\mu)^{k}t^{-1}\cdot kI_k(2\sqrt{\lambda\mu}\,t)$$

Hence, from (20), taking the inverse Laplace transforms, we have

$$P_n(t) = \frac{e^{-(\lambda-\mu)t}}{\lambda}\left[\left(\sqrt{\frac{\lambda}{\mu}}\right)^{n-i+1}\frac{i}{\sum_{m=(i-n)^+}^{\infty}}\frac{n-i+2m+1}{t}\cdot I_{n-i+2m+1}(2\sqrt{\lambda\mu}\,t)\right. \tag{24}$$

$$\left.+\left(\frac{\lambda}{\mu}\right)^{n+1}\frac{\infty}{\sum_{k=n+i+2}}\left(\sqrt{\frac{\mu}{\lambda}}\right)^{k}\cdot\frac{k}{t}\,I_k(2\sqrt{\lambda\mu}\,t)\right]$$

On using the well-known recurrence relation

$$(2\nu/z)I_\nu(z) = I_{\nu-1}(z) - I_{\nu+1}(z) \tag{25}$$

and after some obvious simplification, we have

$$P_n(t) = e^{-(\lambda+\mu)t}\left[(\sqrt{\mu/\lambda})^{i-n}I_{|n-i|} + (\sqrt{\mu/\lambda})^{i-n+1}I_{n+i+1}\right. \tag{26}$$

$$\left.+(1-\lambda/\mu)(\lambda/\mu)^n\sum_{k=n+i+2}^{\infty}(\sqrt{\mu/\lambda})^k I_k\right]$$

where $I_\nu \equiv I_\nu(2\sqrt{\lambda\mu}\,t)$. Again, since for integral ν,

$$I_{-\nu}(z) = I_\nu(z) \tag{27}$$

as can be seen from (22), the $I_{|n-i|}$ in (26) can be replaced by I_{n-i}.

2.3 The Busy Period Distribution

For finding the distribution of busy period, we assume that at time $t = 0$ the system starts with the arrival of a customer which makes the number in the system $n = 1$ and that there is an absorbing barrier at the state $n = 0$. Then, clearly,

$$P_0(t) = P\{\text{busy period} \leqslant t\} = F(t), \text{ say} \tag{1}$$

is the probability distribution function (P.D.F.) of the busy period duration. Under these assumptions, the $P_n(t)$ are given by

$$\frac{d}{dt}P_n(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n = 2, 3, \ldots \tag{2}$$

$$\frac{d}{dt}P_1(t) = -(\lambda+\mu)P_1(t) + \mu P_2(t) \tag{3}$$

$$\frac{d}{dt}P_0(t) = \mu P_1(t) \tag{4}$$

Using the generating function

$$G(z, t) = \sum_{n=1}^{\infty}P_n(t)z^n \tag{5}$$

we have, from (2) and (3),

$$\frac{d}{dt}G(z, t) + (\lambda+\mu-\lambda z-\mu/z)G(z, t) = -\mu P_1(t) \tag{6}$$

Now, taking L.T.'s on both sides of (6), and denoting the transforms by corresponding small letters, we have

$$(s+\lambda+\mu-\lambda z-\mu/z)g(z, s) = z - \mu p_1(s) \tag{7}$$

or,

$$g(z, s) = \frac{z-\mu p_1(s)}{s+\lambda+\mu-\lambda z-\mu/z} \tag{8}$$

The two zeros of the denominator are given by 2.2(10), (11). As in Section 2.2, by Rouché's theorem the numerator of the right-hand member of (8) vanishes at $z = \alpha_1$. Thus,

$$\mu p_1(s) = \alpha_1 \tag{9}$$

From (1), (4) and (9), the busy period density function is

$$f(t) = \frac{d}{dt}F(t) = \mu P_1(t) = \mathcal{L}^{-1}\alpha_1 \tag{10}$$

$$= \mathcal{L}^{-1}\left[\frac{(s+\lambda+\mu)-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}{2\lambda}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2\mu}{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}\right]$$

$$= 2\mu\cdot\frac{1}{2\sqrt{\lambda\mu}}\cdot\frac{1}{t}\cdot I_1(2\sqrt{\lambda\mu}\,t)\cdot e^{-(\lambda+\mu)t}$$

$$= \sqrt{\mu/\lambda}\,e^{-(\lambda+\mu)t}\cdot\frac{1}{t}I_1(2\sqrt{\lambda\mu}\,t)$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind and order 1, and we have used 2.2 (21) for evaluating the inverse Laplace transform.

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The Initial Busy Period

The initial busy period may start with any initial number $n = i$ in the system instead of $n = 1$ as we assumed above. So, proceeding as above and taking $G(z, 0) = z^i$, we see that the busy period density is

$$f_i(t) = \mathscr{L}^{-1}(\alpha_1^i) \tag{11}$$

$$= e^{-(\lambda+\mu)t} \cdot i(\sqrt{\mu/\lambda})^i \cdot \frac{1}{t} I_i(2\sqrt{\lambda\mu}\, t).$$

Alternatively, (11) may be proved as follows.

We know that the busy period duration does not depend on the queue discipline. The initial busy period under consideration starts with i customers in the system; let these customers be numebered 1, 2, ..., i. Let us first serve the customer '1' followed by service of all customers that arrive during his service time, then followed by all customers who arrive during their service times, and so on until all such arrivals have been serviced. Then, we treat similarly customers 2, 3, ..., i. Thus it is clear that the distribution of a busy period starting with i customers is the i-fold convolution of that of an 'ordinary' busy period starting with one customer. Let the corresponding probability density functions (p.d.f.) of these distributions be, respectively, $f_i(t)$ and $f(t)$. Now, we have shown in (10) that

$$\mathscr{L}f(t) = \alpha_1 \tag{12}$$

Therefore, by the convolution theorem of Laplace transforms,

$$\mathscr{L}f_i(t) = \alpha_1^i \tag{13}$$

from which (11) follows.

2.4 The System M/M/∞

The steady-state solution of the service system M/M/∞ was discussed in Section 1.6. The probabilities, $P_n(t)$, for n units in the system at time t for this system are given by

$$\frac{d}{dt} P_n(t) = -(\lambda+n\mu)P_n(t)+\lambda P_{n-1}(t)+(n+1)\mu P_{n+1}(t), \quad (n = 1, 2, \ldots) \tag{1}$$

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t)+\mu P_1(t) \tag{2}$$

Define the generating function

$$G(z, t) = \sum_{n=0}^{\infty} P_n(t)z^n, \quad |z| \leqslant 1 \tag{3}$$

Multiplying (1) by z^n, summing over n and adding (2) as usual, we have

$$\frac{\partial G}{\partial t} = -\sum_{n=0}^{\infty} (\lambda+n\mu)P_n(t)z^n+\lambda \sum_{n=1}^{\infty} P_{n-1}(t)z^n+\mu \sum_{n=0}^{\infty} (n+1)\, P_{n+1}(t)z^n$$

where $G \equiv G(z, t)$

or,

$$\frac{\partial G}{\partial t} = -\lambda G(z, t)-\mu z \frac{\partial G}{\partial z} +\lambda z G(z, t)+\mu \frac{\partial G}{\partial z}$$

or,

$$\frac{\partial G}{\partial t} - \mu(1-z) \frac{\partial G}{\partial z} = -\lambda(1-z)G(z, t) \tag{4}$$

To solve this partial differential equation of Lagrange type, we write the corresponding simultaneous equations

$$\frac{dt}{1} = \frac{dz}{-\mu(1-z)} = \frac{dG}{-\lambda(1-z)G} \tag{5}$$

The solution of the equation obtained from the first two members of the above equation is given by

$$(1-z)e^{-\mu t} = c_1 \tag{6}$$

and the solution of the equation obtained from the last two members is

$$Ge^{-(\lambda/\mu)z} = c_2 \tag{7}$$

where c_1, c_2 are arbitrary constants. Hence, the general solution of (4) is

$$G(z, t) = e^{(\lambda/\mu)z}\phi\{(1-z)e^{-\mu t}\} \tag{8}$$

where ϕ is an arbitrary function.

Now, if the initial number in the system is i, we have the boundary condition

$$G(z, 0) = z^i \tag{9}$$

From (8) and (9), we have

$$z^i = e^{(\lambda/\mu)}\, \phi(1-z)$$

whence

$$\phi(z) = (1-z)^i \exp[-(\lambda/\mu)(1-z)] \tag{10}$$

Using (10) in (8), we have

$$G(z, t) = \exp[(\lambda/\mu)z]\{1-(1-z)e^{-\mu t}\}^i \cdot \exp[-(\lambda/\mu)\{1-(1-z)e^{-\mu t}\}] \tag{11}$$

$$= (q+pz)^i \exp[-(\lambda q/\mu)(1-z)]$$

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where

$$p = e^{-\mu t},\ q = 1-p \tag{12}$$

Taking the coefficient of z^n in (11), we have

$$P_n(t) = \sum_{k=0}^{n} \binom{i}{n-k} \frac{(\lambda/\mu)^k}{k!} \cdot e^{-\lambda q/\mu} p^{n-k} q^{i-n+k} \tag{13}$$

The mean number in the system at time t is given by

$$L = \left.\frac{\partial G}{\partial z}\right|_{z=1} = ip + \frac{\lambda q}{\mu} = \frac{\lambda}{\mu} + \left(i - \frac{\lambda}{\mu}\right) e^{-\mu t} \tag{14}$$

And, the variance V of the number in the system at time t is given by

$$V = \left.\frac{\partial^2 G}{\partial z^2}\right|_{z=1} + L - L^2 \tag{15}$$

which simplifies to

$$\begin{aligned} V &= ipq + \frac{\lambda q}{\mu} \\ &= (\lambda/\mu) + (i - \lambda/\mu)e^{-\mu t} - ie^{-2\mu t} \end{aligned} \tag{16}$$

It may be noted from (11) that $G(z, t)$ is the product of the generating functions of the binomial distribution

$$P(X = k) = \binom{i}{k} q^{i-k} p^k,\ k = 0, 1, \ldots, i \tag{17}$$

and the Poisson distribution

$$P(X = k) = e^{-\lambda q/\mu} (\lambda q/\mu)^k / k!,\ k = 0, 1, 2, \ldots \tag{18}$$

Thus, the distribution of the number in the system at time t is the convolution of these two distributions. This gives us an alternative way of writing (13) by taking the convolution of the two distributions given by (17) and (18). Also, the mean and variance of the number in the system at time t can then be obtained by adding the means and variances of the above binomial and Poisson distributions, which directly gives the results given in (14) and (16).

For the steady-state distribution of the number in system, we let $t \to \infty$ in (11) and obtain

$$G(z) = \lim_{t \to \infty} G(z, t) = \exp\left[-(\lambda/\mu)(1-z)\right] \tag{19}$$

which is the generating function of the Poisson distribution with mean λ/μ, giving the steady-state probabilities

$$P_n = e^{-\lambda/\mu}(\lambda/\mu)^n/n!\ (n = 0, 1, 2, \ldots) \tag{20}$$

which are as derived in 1.6(31).

PROBLEM SET 2

1. From 2.2(26) deduce the steady-state probability P_n by making $t \to \infty$. [From the asymptotic behaviour of $I_k(z)$, we have

$$I_k(z) \sim \exp(z)/\sqrt{2\pi z} \tag{1}$$

which is independent of k. Hence, for large t,

$$\begin{aligned} e^{-(\lambda+\mu)t} I_k(2\sqrt{\lambda\mu}\, t) &\sim \exp[-(\sqrt{\lambda} - \sqrt{\mu})^2 t]/\sqrt{t} \\ &\to 0 \text{ as } t \to \infty \end{aligned} \tag{2}$$

Thus the first two terms on the right-hand side of 2.2(26) vanish. Also, it is well-known that

$$\exp[(x/2)(y + 1/y)] = \sum_{k=-\infty}^{\infty} y^k I_k(x) = \sum_{k=0}^{m-1} y^k I_k(x) + \sum_{k=m}^{\infty} y^k I_k(x) + \sum_{k=1}^{\infty} y^{-k} I_k(x) \tag{3}$$

since $I_{-k}(x) = I_k(x)$ for integral k.

Taking $x = 2\sqrt{\lambda\mu}\, t$, $y = \sqrt{\mu/\lambda}$, and using (1) and the fact that $\lambda/\mu < 1$, we see that the first and the third sums on the right-hand side of (3) vanish as $t \to \infty$. Thus (3) reduces to

$$\sum_{k=m}^{\infty} (\sqrt{\mu/\lambda})^k I_k(2\sqrt{\lambda\mu}\, t) = \exp[(\sqrt{\lambda\mu}\, t)(\sqrt{\mu/\lambda} + \sqrt{\lambda/\mu})] = \exp[(\lambda+\mu)t]$$

Taking $m = n + i + 2$, we see from 2.2(26) that

$$P_n(t) \to (1 - \lambda/\mu)(\lambda/\mu)^n \text{ as } t \to \infty.]$$

2. Obtain the distribution of an ordinary busy period in M/M/1 system and use it to show that the mean duration of a busy period is $1/(\mu - \lambda)$.

[Mean busy period $= \int_0^\infty t f(t) dt$, where $f(t)$ is the busy period density function given by 2.3(10). Use Erdélyi *et al.* (1954a, p. 195, eq. (1)), *viz.*

$$\mathcal{L}[I_\nu(at)] = a^\nu (s^2 - a^2)^{-1/2} (s + \sqrt{s^2 - a^2})^{-\nu}$$

Alternatively, mean busy period $= -\tilde{f}'(0)$, where $\tilde{f}(s) = \mathcal{L}f(t) = \alpha_1$ from 2.3(10).]

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3. Show that the variance of the (ordinary) busy period is $(\lambda+\mu)/(\mu-\lambda)^3$.

[If B stands for the busy period, Var $[B] = E[B^2] - (E[B])^2$, where $E[B] = 1/(\mu-\lambda)$, and $E[B^2] = \int_0^\infty t^2 f(t)dt$, with $f(t)$ given by 2.3(10). Now use Erdélyi *et al.* (1954a), p. 195, Eq. (2), *viz.*

$$\mathcal{L}[t\ I_\nu(at)] = a^\nu(s+\nu\sqrt{s^2-a^2})(s^2-a^2)^{-3/2}(s+\sqrt{s^2-a^2})^{-\nu}.$$

Alternatively, $E[B^2] = \tilde{f}''(0)$, with $\tilde{f}(s)$ given by 2.3(10), which simplifies to $E[B^2] = 2\mu/(\mu-\lambda)^3$.]

4. In an M/M/c queueing system, show that the duration of a busy period defined as the time during which all the c channels are continuously busy has the density function

$$e^{-(\lambda+c\mu)t}t^{-1}\sqrt{c\mu/\lambda}\ I_1(2\sqrt{c\lambda\mu}\ t).$$

Hence show that the mean duration of such a period is $1/(c\mu-\lambda)$.

5. Show that Eq. 2.4(17), (18) give, respectively, the distributions of the customers who are still in the system at time t out of the initial number i in the system at time zero and out of those who arrive in $(0, t]$.

CHAPTER 3

NON-POISSON QUEUES

3.1 Introduction

In this chapter we shall consider queueing systems in which either the distribution of inter-arrival times or that of service times is a general or arbitrary distribution. As was pointed out in Section 1.4, the negative-exponential distribution possesses the Markovian or the forgetfulness property, so that in the case of an M/M/1 queue we do not have to take into account the time since the last arrival or the elapsed service time of the unit in service. This is not the case with queues where one or both of the arrival and service time distributions are different from exponential. It is explained in the next section.

3.2 Imbedded Markov Chain

To be specific, let us first take the case of an M/G/1 queueing system. Here the inter-arrival time distribution is exponential, which has the Markovian property. However, the service time distribution being general, the probability of a service completion in time $(t, t+\Delta t)$ *depends* upon the time for which the service has already been given. To overcome this complication, we study the system not in continuous time but at the discrete time points at which customers depart after service. Between any two consecutive departures, we have to consider only arrivals which are Markovian. Suppose we are considering the queue length process or the number in the system. The queue length at the departure epochs is said to constitute a Markov chain imbedded in the non-Markovian queue length process in continuous time. The departure instants are called *regeneration points* and are such that a knowledge of the state of the process at any regeneration point renders previous history of the process irrelevant.

In the case of the GI/M/1 queue with general arrivals and exponential service times, the arrival epochs are the regeneration points. Clearly for an M/M/1 system, *all* time points are regeneration points and the whole process in continuous time is Markovian.

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3.3 The System M/G/1

Let us consider the queueing system with Poisson arrivals at a mean rate λ, a general service time distribution with density function $b(v)$ and one server. Let t_n be the time of the n-th departure from the system, $n = 1, 2, \ldots$; $t_0 = 0$. Let $N(t)$ denote the number in the system at time t, and let

$$N_n = N(t_n + 0) \tag{1}$$

i.e., N_n denotes the number in the system left behind by the n-th departing customer. As explained in Section 3.2, the discrete-time process $\{N_n\}$ constitutes a Markov chain imbedded in the continuous-time process $\{N(t)\}$. Further, it can be shown that the transition probabilities of this Markov chain, *viz.*

$$p_{ij} = \mathrm{P}\{N_{n+1} = j \mid N_n = i\}, \quad (i, j \geqslant 0,\ n \geqslant 1) \tag{2}$$

are independent of n. Such a Markov chain is said to be time-homogeneous. In fact, from simple probability arguments it follows that

$$p_{0j} = k_j, \quad (j \geqslant 0) \tag{3}$$

$$\begin{aligned} p_{ij} &= k_{j-i+1}, \quad (i \geqslant 1,\ j \geqslant i-1) \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{4}$$

where k_r is the probability of r arrivals during a service time of arbitrary duration and is given by

$$k_r = \int_0^\infty [e^{-\lambda v}(\lambda v)^r / r\,!]\, b(v)\, dv \tag{5}$$

The probability generating function (p.g.f.) of the distribution $\{k_r\}$ is given by

$$\begin{aligned} K(z) = \sum_{r=0}^{\infty} k_r z^r &= \int_0^\infty e^{-\lambda v + \lambda v z} b(v)\, dv \\ &= \bar{b}(\lambda - \lambda z), \quad (|z| < 1) \end{aligned} \tag{6}$$

where

$$\bar{b}(s) = \int_0^\infty e^{-sv} b(v)\, dv = \mathscr{L}[b(v)] \tag{7}$$

is the Laplace transform (L.T.) of the service time probability density function (p.d.f.) $b(v)$ or, equivalently, the Laplace-Stieltjes transform (L.S.T.) of the corresponding probability distribution function (P.D.F.) $B(v)$:

$$\bar{b}(s) = \int_0^\infty e^{-sv}\, dB(v) \tag{8}$$

Clearly, from (6) we have

$$K'(1) = -\lambda \bar{b}'(0) \tag{9}$$

where the primes denote differentiation. Also, from (7),

$$-\bar{b}'(0) = \int_0^\infty v b(v)\, dv = 1/\mu, \text{ say} \tag{10}$$

which is the mean service time. Thus, (9) and (10) yield

$$K'(1) = \lambda/\mu = \rho \tag{11}$$

which is the traffic intensity.

From (3) and (4) it follows that the transition probability matrix

$$\mathbf{P} = (p_{ij}) \tag{12}$$

is given by

$$\mathbf{P} = \begin{array}{c|cccccc} \nearrow & 0 & 1 & 2 & 3 & \ldots & \ldots \\ \hline 0 & k_0 & k_1 & k_2 & k_3 & \ldots & \ldots \\ 1 & k_0 & k_1 & k_2 & k_3 & \ldots & \ldots \\ 2 & 0 & k_0 & k_1 & k_2 & \ldots & \ldots \\ 3 & 0 & 0 & k_0 & k_1 & \ldots & \ldots \\ \cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \cdot & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array} \tag{13}$$

The elements of $\mathbf{P}$ give one-step transition probabilities in the sense that if $\pi^{(0)} = (\pi_0^{(0)}, \pi_1^{(0)}, \pi_2^{(0)}, \ldots)$ be the probability vector giving the probability distribution of N_m, the number in the system just after the departure of any m-th customer, then the probability distribution of N_{m+1} will be given by

$$\pi^{(1)} = \pi^{(0)}\mathbf{P} \tag{14}$$

Similarly, if $\pi^{(n)}$ is the probability vector for N_{m+n} $(n = 1, 2, \ldots)$, we shall have

$$\begin{aligned} &\pi^{(2)} = \pi^{(1)}\mathbf{P} = \pi^{(0)}\mathbf{P}^2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\pi^{(n)} = \pi^{(n-1)}\mathbf{P} = \pi^{(0)}\mathbf{P}^n \end{aligned} \tag{15}$$

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Thus, it is clear that $\mathbf{P}^n$ is the matrix giving the n-step transition probabilities, whose (i,j)-th element, say $p_{ij}^{(n)}$, is

$$p_{ij}^{(n)} = P\{N_{m+n} = j \mid N_m = i\} \tag{16}$$

which is independent of m as discussed earlier.

It can be shown (see, for example, Prabhu (1965a), p. 65, problem 25, or a text book on Markov chains) that when $\rho < 1$, the Markov chain $\{N_n\}$ is *ergodic*, i.e. a steady-state vector

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots) \tag{17}$$

exists, where

$$\pi = \lim_{n \to \infty} \pi^{(n)} \tag{18}$$

Then, using (14), we see that the steady-state probability vector π is given by

$$\pi = \pi \mathbf{P} \tag{19}$$

From (19), we have

$$\pi_j = \pi_0 k_j + \sum_{i=1}^{j+1} \pi_i k_{j-i+1} \quad (j \geqslant 1) \tag{20}$$

Let us define the p.g.f.

$$\Pi(z) = \sum_{j=0}^{\infty} \pi_j z^j \quad (|z| < 1) \tag{21}$$

Multiplying (20) by z^j and summing over j, we have

$$\Pi(z) = \pi_0 \sum_{j=0}^{\infty} k_j z^j + \sum_{j=0}^{\infty} \sum_{i=1}^{j+1} \pi_i k_{j-i+1} z^j \tag{22}$$

$$= \pi_0 K(z) + \sum_{i=1}^{\infty} \sum_{j=i-1}^{\infty} \pi_i k_{j-i+1} z^j$$

changing the order of summation in the second term. Now, we can write the second term as

$$\sum_{i=1}^{\infty} \pi_i z^{i-1} \left(\sum_{j=i-1}^{\infty} k_{j-i+1} z^{j-i+1} \right) = \sum_{i=1}^{\infty} \pi_i z^{i-1} \left(\sum_{r=0}^{\infty} k_r z^r \right) \tag{23}$$

$$= (1/z) \sum_{i=1}^{\infty} \pi_i z^i K(z)$$

$$= \{K(z)/z\}[\Pi(z) - \pi_0]$$

Thus, (22) simplies to

$$\Pi(z) = \pi_0 K(z) + \{K(z)/z\}[\Pi(z) - \pi_0] \tag{34}$$

which gives

$$\Pi(z) = \frac{\pi_0 (1-z) K(z)}{K(z) - z} \tag{25}$$

To find π_0, we use the fact that $\lim_{z \uparrow 1} \Pi(z) = 1$. By l'Hôpital's rule, we have

$$1 = \lim_{z \uparrow 1} \Pi(z) = \pi_0 K(1-0) \lim_{z \uparrow 1} \frac{(-1)}{K'(z) - 1} \tag{26}$$

Now, $K(1-0) = 1$ and from (11), $K'(1) = \rho$; therefore

$$\pi_0 = 1 - \rho \tag{27}$$

and, hence,

$$\Pi(z) = \frac{(1-\rho)(1-z)K(z)}{K(z) - z} \tag{28}$$

(28) gives the generating function $\Pi(z)$ of steady-state probabilities $\{\pi_j\}$, so that π_j is the coefficient of z^j in the expansion of the right-hand member of (28).

Waiting Time

Let N denote the steady-state number in the system left behind by a departing customer and w his total time spent in the system. Then N is the number of arrivals during time w. Thus

$$\pi_j = \mathrm{P}\{N = j\} = \int_0^{\infty} [e^{-\lambda w} (\lambda w)^j / j\,!] f(w) dw \tag{29}$$

where $f(w)$ is the probability density function of w. From (29), on multiplying by z^j and summing over j, we have

$$\Pi(z) = \int_0^{\infty} e^{-\lambda(1-z)w} f(w) dw = \bar{f}(\lambda - \lambda z) \tag{30}$$

where

$$\bar{f}(s) = \int_0^{\infty} e^{-sw} f(w) dw \tag{31}$$

is the L.T. of $f(w)$. Now, let $f_q(\cdot)$ be the p.d.f. of w_q, the queueing time (time spent in queue) of the typical departing customer in steady state and let v be his service time; then,

$$w = w_q + v \tag{32}$$

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and hence

(33) $$f(w) = f_q(w) * b(w)$$

where the asterisk denotes convolution. Using the convolution theorem of Laplace transforms, we have

(34) $$\bar{f}(s) = \bar{f}_q(s)\bar{b}(s)$$

where $\bar{f}_q(s)$ and $\bar{b}(s)$ are the L.T.'s of $f_q(w)$ and $b(w)$, respectively. From (30) and (34) it follows that

(35) $$\Pi(z) = \bar{f}_q(\lambda-\lambda z)\bar{b}(\lambda-\lambda z)$$

or, using (6),

(36) $$\Pi(z) = \bar{f}_q(\lambda-\lambda z)K(z)$$

Now, writing $\lambda-\lambda z = s$ in (36), we have

(37) $$\bar{f}_q(s) = \frac{\Pi\,(1-s/\lambda)}{K(1-s/\lambda)}$$

whence, on using (28) and (6),

(38) $$\bar{f}_q(s) = \frac{(1-\rho)s}{s-\lambda+\lambda\bar{b}(s)}$$

From (28) and (38) we can find the mean number in the system and the mean queueing time. The mean number in the system is

(39) $$E(N) = \Pi'(1) = \rho+\frac{\lambda^2E(v^2)}{2(1-\rho)} = L, \text{ say}$$

and the mean queueing time is

(40) $$E(w_q) = -\bar{f}_q'(0) = \frac{\lambda E(v^2)}{2(1-\rho)} = W_q, \text{ say}$$

where $E(v^2)$ is the expectation of v^2 or the second moment, about the origin, of the service time distribution.

If W be the mean time spent in the system, we have

(41) $$W = W_q+(1/\mu)$$

From (39)–(41), it follows that the two relations

(42) $$L = \lambda W, \quad L_q = \lambda W_q$$

hold for this system also.

In (39) it may be noted that

(43) $$E(v^2) = \sigma_v^2+1/\mu^2$$

where σ_v^2 is the variance of the service time v. Hence (39) may be written

(44) $$L = \rho+\frac{\lambda^2\sigma_v^2+\rho^2}{2(1-\rho)}$$

and, therefore,

(45) $$L_q = \frac{\lambda^2\sigma_v^2+\rho^2}{2(1-\rho)}$$

Deductions for M/M/1

Considering the special case of M/M/1, we have

(46) $$b(v) = \mu e^{-\mu v}$$

and, therefore,

(47) $$\bar{b}(s) = \frac{\mu}{\mu+s}$$

(48) $$K(z) = \bar{b}(\lambda-\lambda z) = \mu/(\mu+\lambda-\lambda z) = (1+\rho-\rho z)^{-1}$$

Hence (28) reduces to

(49) $$\Pi(z) = \frac{1-\rho}{1-\rho z}$$

and (38) simplifies to

(50) $$\bar{f}_q(s) = \frac{(1-\rho)(\mu+s)}{s+\mu-\lambda}$$

(49) gives the well-known M/M/1 system-size distribution [*Cf.* 1.5(17)]:

(51) $$\pi_j = (1-\rho)\rho^j$$

Also, (50) can be written

(52) $$\bar{f}_q(s) = (1-\rho)+(1-\rho).\frac{\lambda}{s+\mu-\lambda}$$

whence, on taking the inverse Laplace transform, we get

(53) $$f_q(w) = (1-\rho)\delta(w)+(1-\rho)\lambda e^{(\lambda-\mu)w}$$

which agrees with 1.5(26), (27). [$\delta(w)$ is the Dirac delta function.]

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Higher moments

(i) The *variance* of N, the number in the system, is given by

$$\text{Var}[N] = E[N^2] - (E[N])^2 \tag{54}$$

where $E[N] = L$ is given by (39). Also, it may be noted that

$$\begin{aligned} \Pi''(1-0) &= \lim_{z \uparrow 1} \frac{d^2\Pi(z)}{dz^2} \\ &= E[N(N-1)] \\ &= E[N^2] - E[N] \end{aligned} \tag{55}$$

is the factorial moment of N, of order 2. To evaluate the limit occurring in (55), we write (28) in the form

$$\Pi(z)\{K(z) - z\} = (1-\rho)(1-z)K(z) \tag{56}$$

Differentiating (56) w.r.t. z thrice, proceeding to limits and simplifying, we have

$$\Pi''(1-0) = \frac{\lambda^2 E(v^2)}{1-\rho} + \frac{\lambda^3 E(v^3)}{3(1-\rho)} + \frac{\lambda^4 (E(v^2))^2}{2(1-\rho)^2} \tag{57}$$

Using (54), (55), we have

$$\text{Var}[N] = \Pi''(1-0) + L - L^2 \tag{58}$$

where L and $\Pi''(1-0)$ are given by (39) and (57).

Considering the particular case of $M/D/1$ with constant service time $1/\mu$, we have

$$E(v^2) = 1/\mu^2,\ E(v^3) = 1/\mu^3 \tag{59}$$

so that

$$\begin{aligned} \Pi''(1-0) &= \frac{\rho^2}{1-\rho} + \frac{\rho^3}{3(1-\rho)} + \frac{\rho^4}{2(1-\rho)^2} \\ &= \frac{\rho^2(\rho^2 - 4\rho + 6)}{6(1-\rho)^2} \end{aligned} \tag{60}$$

Again, *for* $M/M/1$, since

$$E(v^2) = 2/\mu^2,\ E(v^3) = 6/\mu^3 \tag{61}$$

we have

$$\begin{aligned} \Pi''(1-0) &= \frac{2\rho^2}{1-\rho} + \frac{2\rho^3}{1-\rho} + \frac{2\rho^4}{(1-\rho)^2} \\ &= \frac{2\rho^2}{(1-\rho)^2} \end{aligned} \tag{62}$$

which tallies with 1.5 (34).

(ii) The *variance of* w_q, the waiting time in the queue, is given by

$$\text{Var}[w_q] = \bar{f}_q''(0) - (\bar{f}_q'(0))^2 \tag{63}$$

where $\bar{f}_q(s)$ is the Laplace transform of the probability density function of w_q, given by (38) and $-\bar{f}_q'(0) = E[w_q] = W_q$, given by (40).

To evaluate $\bar{f}_q''(0)$, we write (38) as

$$\{s - \lambda + \lambda\bar{b}(s)\}\bar{f}_q'(s) = (1-\rho)s \tag{64}$$

and differentiate w.r.t. s thrice and proceed to limits as $s \to 0$. Thus, on simplification, we have

$$\begin{aligned} \bar{f}_q''(0) &= \frac{\lambda\{3W_q E(v^2) + E(v^3)\}}{3(1-\rho)} \\ &= \frac{L_q E(v^2) + \lambda E(v^3)/3}{1-\rho} \end{aligned} \tag{65}$$

Ex. 1. If $Q(x) = N(x)/D(x)$ be indeterminate of the form $0/0$ at $x = a$, *i.e.*, $N(a) = D(a) = 0$, show that

$$\lim_{x \to a} Q'(x) = \lim_{x \to a} \frac{D'N'' - N'D''}{2D'^2} \tag{1}$$

where the primes (dashes) stand for differentiation w.r.t. x, and the arguments have been suppressed on the right-hand side for simplicity.

We have

$$Q'(x) = \frac{DN' - ND'}{D^2}$$

which is also of the form $0/0$ at $x = a$. Applying the l'Hôpital's rule twice we have the result.

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Eq. (1) above is very important and often used in queueing literature for finding mean queue length, mean waiting time, etc. For example, in Eq. 3.3(28), taking $N = (1-z)K(z)$, $D = K(z)-z$ and using Eq. (1) above, we have

$$\pi'(1) = K'(1) + \frac{K''(1)}{2(1-\rho)}$$

Now, $K'(1) = \rho$, and from 3.3(6), $K''(1) = \lambda^2 E(v^2)$. Hence 3.3(39) follows.

The same method can be used to obtain 3.3(40), and similar results in the following sections.

3.4 The System GI/M/1

Let us consider the single-server queueing system with general inter-arrival time distribution $A(u)$ and exponential service time with mean $1/\mu$. Let t_n be the arrival time of the n-th customer ($n = 1, 2, \ldots$), and let $N(t)$ denote the number in the system at time t. Further, we write

$$N_n = N(t_n - 0) \tag{1}$$

so that N_n is the number in the system just before the n-th arrival. Then, since the service process is Markovian, the set $\{N_n\}$ constitutes a Markov chain imbedded in the process $N(t)$. Let k_r be the probability of r service completions during an inter arrival time of arbitrary duration, given by

$$k_r = \int_0^\infty [e^{-\mu u}(\mu u)^r / r\,!]\, dA(u) \tag{2}$$

Then, the transition probabilities of the Markov chain are

$$p_{ij} = k_{i-j+1} \quad (i \geqslant 0,\ 0 < j \leqslant i+1) \tag{3}$$

$$p_{i0} = \int_0^\infty \int_0^u e^{-\mu v} \frac{\mu^{i+1} v^i}{i\,!}\, dv dA(u) \tag{4}$$

$$= \int_0^\infty dA(u) \left[1 - \sum_{r=0}^{i} e^{-\mu u} \frac{(\mu u)^r}{r\,!} \right], \text{ by successive integration by parts}$$

$$= 1 - \sum_{r=0}^{i} k_r = \alpha_i, \text{ say}$$

It may be noted that

$$\alpha_i = k_{i+1} + k_{i+2} + \ldots \tag{5}$$

That $p_{i0} = \alpha_i$ may also be arrived at as follows. The number of customers in the system just after the arrival becomes $i+1$ and this will drop to zero just before the next arrival if the service facility is capable of completing at least $i+1$ services during an inter-arrival time.

The transition probability matrix $\mathbf{P} = (p_{ij})$ is given by

$$\mathbf{P} = \begin{array}{c|cccccc} \nearrow & 0 & 1 & 2 & 3 & \ldots \\ \hline 0 & \alpha_0 & k_0 & 0 & 0 & \ldots \\ 1 & \alpha_1 & k_1 & k_0 & 0 & \ldots \\ 2 & \alpha_2 & k_2 & k_1 & k_0 & \ldots \\ 3 & \cdot & \cdot & \cdot & \cdot & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \end{array} \tag{6}$$

As in Section 3.3, the steady-state probabilities are given by

$$\pi_0 = \sum_{i=0}^{\infty} \pi_i \alpha_i \tag{7}$$

$$\pi_j = \sum_{i=j-1}^{\infty} \pi_i k_{i-j+1} \quad (j \geqslant 1) \tag{8}$$

As a trial solution of (7), (8) we take

$$\pi_j = ab^j \quad (a > 0,\ 0 < b < 1) \tag{9}$$

Substitution of (9) in (7) gives

$$a = \sum_{i=0}^{\infty} ab^i \left(\sum_{r=i+1}^{\infty} k_r \right) \tag{10}$$

$$= \sum_{r=1}^{\infty} \sum_{i=0}^{r-1} a k_r b^i, \text{ on changing the order of summation}$$

$$= a \sum_{r=0}^{\infty} k_r \left(\frac{1-b^r}{1-b} \right)$$

$$= \frac{a}{1-b} \{1 - K(b)\}$$

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or,

$$K(b) = b \tag{11}$$

where

$$K(z) = \sum_{r=0}^{\infty} k_r z^r \quad (|z| < 1) \tag{12}$$

is the generating function of k_r. And, substitution of (9) in (8) gives

$$ab^j = \sum_{i=j-1}^{\infty} ab^i k_{i-j+1} \tag{13}$$

$$= ab^{j-1} \sum_{r=0}^{\infty} k_r b^r = ab^{j-1} K(b)$$

which also implies that $K(b) = b$. Thus we see that (9) is a solution of (7) and (8) if b is a root of the equation

$$K(z) = z \tag{14}$$

i.e., b is a root, less than 1, of the equation

$$\tilde{a}(\mu - \mu z) = z \tag{15}$$

as can be seen from (2) and (12), where $\tilde{a}(s)$ is the L.T. of inter-arrival time density function $a(u)$. Evaluating a from the normalizing condition

$$\sum_{j=0}^{\infty} \pi_j = 1 \tag{16}$$

we obtain the geometric distribution

$$\pi_j = (1-b)b^j \tag{17}$$

where b is as given above.

It remains to be shown that the equation $K(z) = z$ has a root b such that $0 < b < 1$, which we do in the following

Theorem. Let $K(z) = \sum_{r=0}^{\infty} k_r z^r$ be a p.g.f., and assume that $k_0 > 0$, $0 < K'(1) = \alpha < \infty$. Then the equation $K(z) = z$ has a root z_0 such that $0 < z_0 < 1$ if and only if $\alpha > 1$. [See Feller (1968); Prabhu (1965a), p. 63.]

Let us consider the function

$$f(z) = K(z)/z = k_0/z + k_1 + k_2 z + \ldots \quad (0 < z \leqslant 1) \tag{18}$$

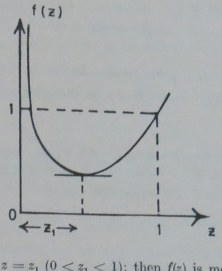
We have

$$f(+0) = \infty, \quad f(1) = 1 \tag{19}$$

Also,

$$f''(z) = 2k_0 z^{-3} + 2k_3 + 6k_4 z + \ldots > 0 \tag{20}$$

Therefore, $f'(z)$ is a monotonic increasing function and vanishes at most once in $0 < z < 1$ if and only if $f'(1) = \alpha - 1 > 0$, since we have $f'(0) = -\infty$.



Let $f'(z) = 0$ at $z = z_1$ $(0 < z_1 < 1)$; then $f(z)$ is monotonic decreasing in $(0, z_1)$ and monotonic increasing in $(z_1, 1]$. Since $f(1) = 1$, it follows that $f(z_1) < 1$ and hence there exists exactly one value z_0 $(0 < z_0 < z_1)$ such that $f(z_0) = 1$, or $K(z_0) = z_0$. We can *a fortiori* assert that $0 < z_0 < 1$. This completes the proof of the theorem.

Thus, we see that (17) determines the steady-state probabilities π_j *uniquely*.

Queueing Time. Using (17) and proceeding as in Section 1.5, we can obtain the distribution of the queueing time w_q of an arriving customer in the steady state as follows.

$$P(w_q = 0) = \pi_0 = 1 - b \tag{21}$$

$$P(w < w_q < w + dw) = \sum_{j=1}^{\infty} (1-b)b^j \cdot \frac{(\mu w)^{j-1} e^{-\mu w}}{(j-1)!} \cdot \mu dw \tag{22}$$

$$= (1-b)b\mu e^{-(1-b)\mu w}\, dw$$

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The relations (17) and (21), (22) are important in that they show that the system size distribution is geometric and the queueing time distribution is truncated exponential irrespective of the form of the interarrival time distribution, just as they were in the case of M/M/1 queue, with, of course, different parameters, ρ being replaced by b, the unique root in $0 < z < 1$ of the equation $\tilde{a}(\mu - \mu z) = z$.

For the M/M/1 system, we have

$$\tilde{a}(s) = \lambda/(\lambda + s) \tag{23}$$

Therefore, b is the root of

$$\lambda/(\lambda + \mu - \mu z) = z \tag{24}$$

or,

$$\mu z^2 - (\lambda + \mu)z + \lambda = 0 \tag{25}$$

whose roots are 1 and ρ. Hence $b = \rho$, as discussed above.

3.5 GI/M/c

The method of Section 3.4 is applicable to the more general case of the multichannel system GI/M/c. As before, we shall have the one-step transition probabilities between two consecutive arrivals as

$$p_{ij} = k_{i-j+1} \; (i \geqslant 0, \; 0 < j \leqslant i+1) \tag{1}$$

where k_r is the probability that r services are completed during an inter-arrival time of arbitrary duration. Now, as long as all the c servers are busy, which requires that $i \geqslant c-1$, we have

$$k_r = \int_0^\infty \frac{e^{-c\mu u} (c\mu u)^r}{r\,!} \, dA(u) \tag{2}$$

The steady-state probabilities π_j are given by

$$\pi_j = \sum_{i=j-1}^{\infty} \pi_i p_{ij} \tag{3}$$

Thus, we have to solve (3), where the p_{ij} are given by (1) and (2). As a trial solution (suggested by the solution for GI/M/1), let us write

$$\pi_j = C\theta^j, \quad 0 < \theta < 1, \; j \geqslant c-1, \quad C \text{ a constant.} \tag{4}$$

Substitution from (4) in (3) yields

$$\theta = \sum_{i=j-1}^{\infty} \theta^{i-j+1} k_{i-j+1} = \sum_{r=0}^{\infty} \theta^r k_r \tag{5}$$

Now, using (2) and simplifying, we have

$$\theta = \tilde{a}(c\mu - c\mu\theta) \tag{6}$$

where, as before, $\tilde{a}(s)$ is the Laplace-Stieltjes transform of $A(u)$. Now, as in Section 3.4, it can be shown that (6) possess a unique root in $0 < \theta < 1$; let us call it b. Then, for $\theta = b$, the values of π_j $(j \geqslant c-1)$ are given by (4).

Thus, we have

$$\pi_j = Cb^j \; (j \geqslant c-1) \tag{7}$$

The evaluation of C and the determination of π_j for $j < c-1$ are rather complicated and will be discussed later only for the case $c = 2$. We give below some interesting results which can be obtained without evaluation of C.

The conditional probability that there are j units in the system, given that the customers have to queue for service, is

$$\pi_j \Big/ \sum_{r=c}^{\infty} \pi_r = (1-b)b^{j-c}, \; (j \geqslant c) \tag{8}$$

Therefore, the conditional queueing time distribution of the customers who have to queue up has a density function $f_q(w)$ given by

$$f_q(w)dw = \sum_{j=c}^{\infty} (1-b)b^{j-c} \cdot \frac{(c\mu w)^{j-c} e^{-c\mu w}}{(j-c)\,!} \cdot c\mu dw \tag{9}$$

$$= (1-b)c\mu \exp[-(1-b)c\mu w]dw$$

(8) and (9) prove the interesting results that the conditional distributions of the number in the system for customers who have to queue and of their queueing time are, respectively, the geometric and the exponential distributions for the GI/M/c system as they were for M/M/1.

Note that in writing equation (9) we have used the fact that if there are j units in the system, *i.e.* $j-c$ in the queue, an arriving customer has to wait in queue until $j-c+1$ services are completed.

3.6 The Supplementary Variable Technique

Besides the imbedded Markov chain method, another way to analyze the non-Poisson queues is to use one or more extra variables, called supplementary variables in the definition of states of the system, so that the process with the new definition becomes Markovian. For the M/G/1 or GI/M/1 system we need only one supplementary variable, being the time since the last departure or last arrival, respectively. The most general queueing

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system GI/G/c can thus be studied by using $c+1$ supplementary variables, one for the time since the last arrival and c for the elapsed service time for each of the c servers. We illustrate the method below for the system M/G/1. The supplementary variable method has the advantage over the imbedded Markov chain method that here we can study the system in continuous time instead of at discrete time points.

The system M/G/1

Let $P_n(x, t)dx$, $(n \geqslant 0)$, be the joint probability that at time t there are n units waiting in the queue and the elapsed service time of the unit in service lies in the interval $(x, x+dx)$. Also, let $P_{00}(t)$ be the probability that the system is empty at time t. For the steady-state, we consider the limiting probability density

$$P_n(x) = \lim_{t \to \infty} P_n(x, t), \ (n \geqslant 0) \tag{1}$$

and the limiting probability

$$P_{00} = \lim_{t \to \infty} P_{00}(t) \tag{2}$$

Let the arrivals be Poisson distributed with mean rate λ. Also, let $\eta(x)$ be the conditional probability density of a service completion, given that the elapsed service time is x, so that

$$\eta(x) = \frac{b(x)}{1-B(x)} \tag{3}$$

where $B(v)$ and $b(v)$ are, respectively, the distribution function and the density function of the service time. From (3), we have, on integration,

$$B(v) = 1-e^{-\int_0^v \eta(x)dx} \tag{4}$$

and, therefore,

$$b(v) = \eta(v)e^{-\int_0^v \eta(x)dx} \tag{5}$$

Arguing as in Section 1.5, we have the following transition equations.

$$P_n(x+\Delta x) = P_n(x)[1-\{\lambda+\eta(x)\}\Delta x]+P_{n-1}(x).\lambda\Delta x, \ (n \geqslant 1) \tag{6}$$

$$P_0(x+\Delta x) = P_0(x)[1-\{\lambda+\eta(x)\}\Delta x] \tag{7}$$

$$P_{00} = P_{00}(1-\lambda\Delta x)+\int_0^\infty \eta(x)\Delta x.\ P_0(x)dx \tag{8}$$

From (6)–(8), we have

$$\frac{d}{dx}P_n(x) = -[\lambda+\eta(x)]P_n(x)+\lambda P_{n-1}(x), \ (n \geqslant 1) \tag{9}$$

$$\frac{d}{dx}\ P_0(x) = -[\lambda+\eta(x)]P_0(x) \tag{10}$$

$$\lambda P_{00} = \int_0^\infty P_0(x)\eta(x)dx \tag{11}$$

These equations are to be solved with the following boundary conditions :

$$P_n(0) = \int_0^\infty P_{n+1}(x)\eta(x)dx, \ (n \geqslant 1) \tag{12}$$

$$P_0(0) = \int_0^\infty P_1(x)\eta(x)dx+\lambda P_{00} \tag{13}$$

and the normalizing condition

$$P_{00}+\sum_{n=0}^{\infty}\int_0^\infty P_n(x)dx = 1 \tag{14}$$

Obviously, the probability P_n of finding n persons in the queue, irrespective of the value of x, is given by

$$P_n = \int_0^\infty P_n(x)dx, \ (n \geqslant 1) \tag{15}$$

$$P_0 = P_{00}+\int_0^\infty P_0(x)dx. \tag{16}$$

Define the generating functions

$$P_q(z, x) = \sum_{n=0}^{\infty} P_n(x)z^n, \ |z| \leqslant 1 \tag{17}$$

$$P_q(z) = \int_0^\infty P_q(z, x)dx \tag{18}$$

$$P(z) = P_q(z)+P_{00} \tag{19}$$

Multiplying (9) by z^n, summing over n, and adding (10), we have

$$\frac{d}{dx}\ P_q(z, x)+[\lambda(1-z)+\eta(x)]\ P_q(z, x) = 0 \tag{20}$$

whose solution is

$$P_q(z, x) = P_q(z, 0)\exp\left[-\lambda(1-z)x-\int_0^x \eta(v)dv\right]$$

$$= P_q(z, 0)\exp\left[-\lambda(1-z)x\right]\{1-B(x)\}. \tag{21}$$

Hence,

$$P_q(z) = P_q(z, 0)\cdot\ \frac{1-\bar{b}(\lambda-\lambda z)}{\lambda(1-z)} \tag{22}$$

where we have used the fact that

$$\int_0^\infty e^{-sx}\ B(x)dx = \int_0^\infty e^{-sx}\left(\int_0^x b(v)dv\right)dx = \bar{b}(s)/s \tag{23}$$

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To evaluate $P_q(z, 0)$, we have, from (12) and (13), on multiplying by appropriate powers of z and adding,

$$P_q(z, 0) = \frac{1}{z}\int_0^\infty \sum_{n=1}^{\infty} P_n(x)\eta(x)z^n dx + \lambda P_{00} \tag{24}$$

$$= \lambda P_{00}\left(1-\frac{1}{z}\right) + \frac{1}{z}\int_0^\infty P_q(z, x)\eta(x)dx$$

on using (11). Now, substituting from $P_q(z, x)$ from (21) in (24), evaluating the integral, we see that $P_q(z, 0)$ is given by

$$P_q(z, 0) = \frac{\lambda P_{00}(1-z)}{\bar{b}(\lambda-\lambda z)-z} \tag{25}$$

From (22) and (25), we have

$$P_q(z) = \frac{P_{00}[1-\bar{b}(\lambda-\lambda z)]}{\bar{b}(\lambda-\lambda z)-z} \tag{26}$$

Hence (19) gives

$$P(z) = \frac{P_{00}[1-z]}{\bar{b}(\lambda-\lambda z)-z} \tag{27}$$

Since $P(1-0) = 1$, we have, using l'Hôpital's rule,

$$P_{00} = 1-\rho \tag{28}$$

where ρ is the traffic intensity $= -\lambda\bar{b}'(0)$. Therefore,

$$P(z) = \frac{(1-\rho)(1-z)}{\bar{b}(\lambda-\lambda z)-z} \tag{29}$$

We proceed to find the relation between the random point state probabilities P_n studied above and the imbedded Markov chain probabilities of Section 3.3. Let P_n^+ be the steady-state probability that there are n units in the system just after a departure; then, as in (11), we have

$$\lambda P_n^+ = \int_0^\infty P_n(x)\eta(x)dx, \quad (n \geqq 0) \tag{30}$$

Using a probability generating function

$$P^+(z) = \sum_{n=0}^{\infty} P_n^+ z^n$$

we have

$$\lambda P^+(z) = \int_0^\infty \sum_{n=0}^{\infty} z^n P_n(x)\eta(x)dx \tag{31}$$

$$= \int_0^\infty P_q(z, x)\eta(x)dx$$

$$= P_q(z, 0)\bar{b}(\lambda-\lambda z)$$

on using (21) and integrating. Now, using (25) and (28) in (31), we have

$$P^+(z) = \frac{(1-\rho)(1-z)\bar{b}(\lambda-\lambda z)}{\bar{b}(\lambda-\lambda z)-z} \tag{32}$$

It may be noted that P_n^+ is the same as the π_n of Section 3.3 and that Eq. (32) above is the same as Eq. 3.3(28).

Clearly, the generating function of P_n, the random epoch probabilities, is $zP_q(z)+P_{00} = P^+(z)$. Hence,

$$P_n^+ = P_n, \; n = 0,1,2,\ldots \tag{33}$$

From (29), by differentiating $P(z)$ w.r.t. z and letting $z \to 1-0$, we have

$$L_q = P'(1-0) = \frac{\lambda^2\bar{b}'(0)}{2(1-\rho)} = \frac{\lambda^2 E(v^2)}{2(1-\rho)}$$

which is the mean number in the queue at a random epoch. If we write L_q^+ for the mean number in the queue at departure instants, we can similarly see from (32), or as already proved in 3.3(45) that

$$L_q^+ = L_q = \frac{\lambda^2 E(v^2)}{2(1-\rho)} = \frac{\rho^2+\lambda^2\sigma_v^2}{2(1-\rho)} \tag{34}$$

Particular cases

(a) M/D/1. Here $\sigma_v = 0$ and hence

$$L_q = \frac{\rho^2}{2(1-\rho)} \tag{35}$$

which is half the mean queue length for M/M/1.

(b) M/E$_k$/1. Here we have, using 1.4(18),

$$E(v^2) = \int_0^\infty v^2[(k\mu)^k v^{k-1}e^{-k\mu v}/(k-1)!]\,dv = \frac{k+1}{k\mu^2} \tag{36}$$

Therefore,

$$L_q = \frac{k+1}{k\mu^2}\cdot\frac{\lambda^2}{2(1-\rho)} = \frac{k+1}{2k}\cdot\frac{\rho^2}{1-\rho} = \frac{k+1}{2k}\cdot\frac{\lambda^2}{\mu(\mu-\lambda)} \tag{37}$$

It can be seen that (35) can be deduced from (37) by letting $k \to \infty$, (see 1.4(19)). Also, setting $k = 1$ in (37), we obtain L_q for M/M/1.

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Having obtained L_q, we can obtain L, W_q, W using the following relations, which hold for most queueing systems (see 1.5(21), (23)) :

$$W_q = L_q/\lambda \tag{38}$$

$$W = W_q + \frac{1}{\mu} \tag{39}$$

$$L = L_q + \rho \tag{40}$$

3.7 The System M/D/c

Let us consider the c-channel system with Poisson arrivals at a mean rate λ and constant service time equal to b. Let $P_n(t)$ be the probability that there are n units in the system at time t; then we have

$$P_n(t+b) = \sum_{i=0}^{c} P_i(t) e^{-\lambda b}[(\lambda b)^n/n\,!] + \sum_{i=c+1}^{n+c} P_i(t) e^{-\lambda b}[(\lambda b)^{n-i+c}/(n-i+c)\,!] \tag{1}$$

Let us assume that the steady-state probabilities

$$P_n = \lim_{t \to \infty} P_n(t) \tag{2}$$

exist; then from (1) we have

$$P_n = (P_0+P_1+\ldots+P_c)k_n + \sum_{i=c+1}^{n+c} P_i k_{n-i+c} \tag{3}$$

where k_r is as defined in 3.3(5) and simplifies to the Poisson probability function with mean λb.

Using the generating functions

$$P(z) = \sum_{n=0}^{\infty} P_n z^n, \quad |z| < 1 \tag{4}$$

$$K(z) = \sum_{n=0}^{\infty} k_n z^n = e^{-\lambda b(1-z)} \tag{5}$$

we have, from (3),

$$\mathrm{P}(z) = \frac{\sum_{i=0}^{c-1} \mathrm{P}_i(z^i - z^c)}{1 - z^c/K(z)} \tag{6}$$

which can be written in the form

$$\mathrm{P}(z) = \frac{(1-z)\sum_{i=0}^{c-1} Q_i z^i}{1 - z^c e^{\lambda b(1-z)}} \tag{7}$$

where

$$Q_r = \mathrm{P}_0 + \mathrm{P}_1 + \ldots + \mathrm{P}_r \quad (r \geqslant 0) \tag{8}$$

Now, we have

$$1 = \lim_{z \to 1-0} \mathrm{P}(z) = \frac{\sum_{i=0}^{c-1} Q_i}{c - \lambda b} \tag{9}$$

so that

$$\sum_{i=0}^{c-1} Q_i = c - \lambda b \tag{10}$$

Thus we should have

$$\rho = \lambda b/c < 1 \tag{11}$$

which is, therefore, the condition for the existence of the steady state.

Now, for sufficiently small positive δ, we have, at $|z| = 1+\delta$,

$$|e^{-\lambda b(1-z)}| \leqslant e^{-\lambda b + \lambda b|z|} = e^{\lambda b \delta} < (1+\delta)^c = |z|^c \tag{12}$$

since $\lambda b < c$, from (11). Hence, by Rouché's theorem, the denominator of (7) has exactly c zeros inside the circle $|z| = 1+\delta$, which we denote by $z_1, z_2, \ldots, z_{c-1}, z_c = 1$. Since the numerator of (7) must also vanish at these values of z, it follows that the equation

$$\sum_{i=0}^{c-1} Q_i z^i = 0 \tag{13}$$

has the $c-1$ roots $z_1, z_2, \ldots, z_{c-1}$. It can be shown that z_r $(r = 1, \ldots, c-1)$ are all distinct. Thus, we have $c-1$ equations and (10) to determine the c unknowns Q_i $(i = 0, 1, \ldots, c-1)$. So the P(z), given by (7) or (6), is completely determined.

As a particular case, for the system M/D/1, we have on setting $c = 1$ in (6),

$$P(z) = \frac{P_0(1-z)}{1 - z e^{\lambda b(1-z)}} \tag{14}$$

where

$$P_0 = 1 - \lambda b = 1 - \rho \tag{15}$$

Thus,

$$\mathrm{P}(z) = (1-\rho)(1-z)/[1 - z e^{\rho(1-z)}] \tag{16}$$

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It may be noted that (16) agrees with 3.6 (32), as could be expected. From (16) the mean queue length and waiting time can be found as before. Thus, the mean number in the system is

$$L = P'(1-0) = \frac{2\rho-\rho^2}{2(1-\rho)} = \rho + \frac{\rho^2}{2(1-\rho)} \tag{17}$$

$$L_q = L-\rho = \frac{\rho^2}{2(1-\rho)} \tag{18}$$

$$W = L/\lambda \tag{19}$$

$$W_q = L_q/\lambda \tag{20}$$

3.8 The System M/E_k/1

The single-server queue with Poisson arrivals and k-Erlang service time distribution admits of simple analysis as follows. Let λ and μ be the mean arrival and service rates, respectively. Then (Cf. Section 1.4(c)), each service time may be considered as made up of k phases, where the time spent in each phase is distributed according to an identical negative exponential distribution with mean $1/k\mu$, so that the mean service time, being the time spent in all the k phases is, $1/\mu$. The phases are numbered in the reverse order, so that the service starts with phase k, after finishing which the phase $k-1$ begins, and so on until the service ends at the completion of phase 1. [Equivalently the phases could be numbered in direct order : 1, 2, ..., k.] Some authors further generalize the model by introducing a variable number of phases, so that an arriving customer may require r phases with probability C_r ($r = 1, 2, \ldots$). In such a case, numbering the phases in reverse order is particularly suitable, so that the service starts at phase r and stops on completion of phase 1.

Let $P_{nr}(t)$ be the probability that at time t there are n units in the *queue* and the unit being served is in phase r of service, and let $P_{00}(t)$ be the corresponding probability that the system is empty. Then the transition equations are (ignoring the $o(\Delta t)$ terms)

$$P_{nr}(t+\Delta t) = P_{nr}(t)\{1-(\lambda+k\mu)\Delta t\}+P_{n-1,\,r}(t)\cdot\lambda\Delta t + P_{n,\,r+1}(t)\cdot k\mu\Delta t,\ (n \geqslant 1;\ 1 \leqslant r < k) \tag{1}$$

$$P_{nk}(t+\Delta t) = P_{nk}(t)\{1-(\lambda+k\mu)\Delta t\}+P_{n-1,\,k}(t)\cdot\lambda\Delta t + P_{n+1,\,1}(t)k\mu\Delta t,\ (n \geqslant 1) \tag{2}$$

$$P_{0r}(t+\Delta t) = P_{0r}(t)\{1-(\lambda+k\mu)\Delta t\}+P_{0,\,r+1}(t)\cdot k\mu\Delta t,\ (1 \leqslant r < k) \tag{3}$$

$$P_{0k}(t+\Delta t) = P_{0k}(t)\{1-(\lambda+k\mu)\Delta t\}+P_{00}(t)\cdot\lambda\Delta t+P_{11}(t)\cdot k\mu\Delta t \tag{4}$$

$$P_{00}(t+\Delta t) = P_{00}(t)(1-\lambda\Delta t)+P_{01}(t)\cdot k\mu\Delta t \tag{5}$$

From the above equations, we have the following for the steady-state probabilities P_{nr}, etc.,

$$0 = -(\lambda+k\mu)P_{nr}+\lambda P_{n-1,\,r}+k\mu P_{n,\,r+1} \tag{6}$$

$$0 = -(\lambda+k\mu)P_{nk}+\lambda P_{n-1,\,k}+k\mu P_{n+1,\,1} \tag{7}$$

$$0 = -(\lambda+k\mu)P_{0r}+k\mu P_{0,\,r+1} \tag{8}$$

$$0 = -(\lambda+k\mu)P_{0k}+\lambda P_{00}+k\mu P_{11} \tag{9}$$

$$0 = -\lambda P_{00}+k\mu P_{01} \tag{10}$$

Let us define the generating functions

$$F_r(x) = \sum_{n=0}^{\infty} P_{nr}x^n, \quad |x| \leqslant 1 \tag{11}$$

$$G(x, y) = \sum_{r=1}^{k} F_r(x)y^r = \sum_{n=0}^{\infty}\sum_{r=1}^{k} P_{nr}x^n y^r, \ |y| \leqslant 1 \tag{12}$$

Multiplying eq. (6)–(9) by appropriate powers of x and y, adding and using (11), (12), we have

$$0 = -(\lambda+k\mu)G(x, y)+\lambda xG(x, y)+(k\mu/y)[G(x, y)-yF_1(x)] + \lambda y^k P_{00}+(k\mu y^k/x)[F_1(x)-P_{01}] \tag{13}$$

or, simplifying and using (10),

$$(\lambda+k\mu-\lambda x-k\mu/y)G(x, y) = k\mu(y^k/x-1)F_1(x)+\lambda y^k(1-1/x)P_{00} \tag{14}$$

To evaluate $F_1(x)$, we put

$$y = \frac{k\mu}{\lambda+k\mu-\lambda x} = y_x, \text{ say,}$$

in (14), which gives

$$F_1(x) = \frac{\lambda(1-x)P_{00}}{k\mu(1-x/y_x^k)} = \frac{\lambda(1-x)P_{00}}{k\mu\left[1-x\left(\frac{\lambda+k\mu-\lambda x}{k\mu}\right)^k\right]} \tag{15}$$

Substituting for $F_1(x)$ from (15) in (14), setting $y = 1$ and simplifying, we have

$$G(x, 1) = \frac{P_{00}\left[\left(\frac{\lambda+k\mu-\lambda x}{k\mu}\right)^k - 1\right]}{1-x\left(\frac{\lambda+k\mu-\lambda x}{k\mu}\right)^k} \tag{16}$$

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Clearly,

$$G(x, 1) = \sum_{n=0}^{\infty} P_n x^n, \quad |x| \leqslant 1 \tag{17}$$

is the generating function of the probabilities P_n of finding n in the queue, irrespective of the service phase. From (16), taking limits and using l'Hôpital's rule, we have

$$\lim_{x \uparrow 1} G(x, 1) = \frac{P_{00}\rho}{1-\rho} \tag{18}$$

where $\rho = \lambda/\mu$. Also, from (17), we have

$$\lim_{x \uparrow 1} G(x, 1) = 1 - P_{00} \tag{19}$$

From (18) and (19) we have, as usual,

$$P_{00} = 1 - \rho \tag{20}$$

Substituting for P_{00} in (16), we have the generating function of the *queue length* probabilities. From (16), differentiating w.r.t. x and taking limits as $x \to 1-0$, we have the mean number in the queue

$$L_q = G'(1-0, 1) = \frac{k+1}{2k} \cdot \frac{\rho^2}{1-\rho} \tag{21}$$

Using eq. 3.6(38)–(40), we can find W_q, W and L.

It may be noted that (16) agrees with 3.6(26), since in the present model the Laplace transform of the service time density is

$$\tilde{b}(s) = \left(\frac{k\mu}{k\mu+s}\right)^k \tag{22}$$

Special Cases. Taking $k = 1$ in (21), we have the well-known result for M/M/1

$$L_q = \frac{\rho^2}{1-\rho} \tag{23}$$

And, letting $k \to \infty$, we have for M/D/1

$$L_q = \frac{1}{2} \cdot \frac{\rho^2}{1-\rho} \tag{24}$$

which agrees with 3.7(18).

Direct Derivation. We give below a method of obtaining W_q, W and L directly.

We know that P_n is the probability that there are n units in the queue and, therefore, $n+1$ in the system ($n = 0, 1, 2, \ldots$). Hence,

$$\begin{aligned} L &= \sum_{n=0}^{\infty} (n+1)P_n \\ &= \sum_{n=0}^{\infty} nP_n + \sum_{0}^{\infty} P_n \\ &= L_q + (1-P_{00}) = L_q + \rho, \quad \text{using (20)} \\ &= \rho + \frac{k+1}{2k} \cdot \frac{\rho^2}{1-\rho}, \quad \text{using (21)} \end{aligned} \tag{25}$$

Also, P_{nr} is the probability that there are $(nk+r)$ *phases* of service in the system (yet to be completed). Hence, the expected number of phases in the system, say L_p, is given by

$$\begin{aligned} L_p &= \sum_{n=0}^{\infty} \sum_{r=1}^{k} (nk+r)P_{nr} \\ &= k \sum_{n} \sum_{r} nP_{nr} + \sum_{n} \sum_{r} rP_{nr} \\ &= k \left. \frac{d}{dx} G(x, 1) \right|_{x=1} + \left. \frac{d}{dy} G(1, y) \right|_{y=1} \end{aligned} \tag{26}$$

From (21), it is clear that the first term on the right-hand side of (26) is kL_q. For evaluating the second term, we have, taking $x = 1$ in (14),

$$G(1, y) = \frac{(y^k-1)F_1(1)}{1-1/y} \tag{27}$$

and, letting $x \to 1$ in (15), we have

$$F_1(1) = \lambda/k\mu \tag{28}$$

Therefore, we have

$$G(1, y) = (y^k + y^{k-1} + \ldots + y)(\lambda/k\mu) \tag{29}$$

Hence,

$$\left. \frac{d}{dy} G(1, y) \right|_{y=1} = \frac{k(k+1)}{2} \cdot \frac{\lambda}{k\mu} = \frac{k+1}{2} \cdot \rho \tag{30}$$

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Thus, (26) simplifies to

$$L_p = k \cdot \frac{k+1}{2k} \cdot \frac{\rho^2}{1-\rho} + \frac{k+1}{2} \cdot \rho \tag{31}$$

$$= \frac{k+1}{2} \cdot \frac{\rho}{1-\rho}$$

Since each phase occupies an average time $1/k\mu$, the mean time spent in the queue is

$$W_q = L_p/k\mu \tag{32}$$

$$= \frac{k+1}{2k} \cdot \frac{\rho}{\mu(1-\rho)} = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)}$$

Again, since the incoming customer adds k phases to the system, the mean time in the system is

$$W = (L_p+k)/k\mu \tag{33}$$

$$= W_q + (1/\mu)$$

3.9 Hyper-Exponential Service Times

It may be noted that the variance of a k-Erlang distribution with mean $1/\mu$ is equal to $1/k\mu^2$ and is thus, for $k > 1$, less than the variance $1/\mu^2$ of the corresponding exponential distribution with the same mean. Thus an Erlang distribution is not useful to approximate a distribution which for a given mean has a variance greater than that of the corresponding exponential distribution. For studying such distributions, consider two channels arranged in parallel (instead of the series-arrangement of Erlang distributions). Let the service times at the two channels be exponentially distributed with means $1/(2\sigma\mu)$ and $1/\{2(1-\sigma)\mu\}$, and let the customers be served at the two channels with probabilities σ and $1-\sigma$, respectively, $(0 < \sigma < 1)$. Without loss of generality we may assume $\sigma \leqslant 1/2$. Then, the overall mean service time is

$$\sigma \cdot \frac{1}{2\sigma\mu} + (1-\sigma) \cdot \frac{1}{2(1-\sigma)\mu} = \frac{1}{\mu} \tag{1}$$

and the overall variance is

$$\int_0^\infty x^2[\sigma.2\sigma\mu e^{-2\sigma\mu x} + (1-\sigma).2(1-\sigma)\mu e^{-2(1-\sigma)\mu x}]dx - (1/\mu^2) \tag{2}$$

$$= \left[\frac{1}{2\sigma(1-\sigma)} - 1\right] \cdot \frac{1}{\mu^2}$$

which has its least value $1/\mu^2$ when $\sigma = 1/2$, and is, therefore, always greater than $1/\mu^2$ for all other values of σ in $(0, 1)$.

Thus, the above distribution, called hyper-exponential distribution, is suitable for approximating distributions which have a greater variance than the exponential with the same mean. Similarly, we can have a hyper-exponential inter-arrival time distribution.

It is easy to see that for the single-server queue with Poisson arrivals and hyper-exponential service times, (which may be denoted by M/HE/1), we have, using 3.3 (45),

$$L_q = \lambda^2 E(v^2)/[2(1-\rho)] = \rho^2/[4\sigma(1-\sigma)(1-\rho)] \tag{3}$$

which is greater than the value $\rho^2/(1-\rho)$ for the M/M/1 queue with the same parameters λ and μ expect when $\sigma = 1/2$.

The values of L, W, W_q can be deduced from (3) by using Eq. 3.3(41), (42).

Ex. 2. A hospital clinic has a doctor examining every patient brought in for a general check-up. The doctor averages 4 minutes on each phase of the check-up although the distribution of time spent on each phase is approximately exponential. If each patient goes through four phases in the check-up and if the arrivals of the patients to the doctor's office are approximately Poisson at the average rate of three per hour, what is the average time spent by a patient waiting in the doctor's office ? What is the average time spent in the examination ? What is the most probable time spent in the examination ? [By permission from Sasieni, Yaspan and Friedman (1959)]

This is an $M/E_4/1$ system with $\lambda = 3$ per hour, and $1/\mu$ = total service time $= 4\times4$ minutes $= 16$ min. $= 4/15$ hour, so that $\mu = 15/4$ per hour. Hence, using 3.8(32), we have

average time spent waiting in doctor's office

$$= W_q = \frac{4+1}{2\times4} \cdot \frac{3}{(15/4)(3/4)} = \frac{5}{8} \times \frac{16}{15} \text{ hour} = 2/3 \text{ hr.} = 40 \text{ min.}$$

Average time spent in examination (service) $= 1/\mu = 16$ min.

Most probable time for examination

$=$ modal value $= (1-1/k)(1/\mu) = (3/4)(16) = 12$ min.

(See Problem Set 1, Prob. 7.)

Ex. 3. At a certain airport it takes exactly 5 minutes to land an aeroplane, once it is given the signal to land. Although incoming planes have scheduled arrival times, the wide variability in arrival times produces an effect which makes the incoming planes appear to arrive in a Poisson fashion at an average rate of 6 per hour. This produces occasional stack-ups at the airport which can be dangerous and costly. Under these circumstances, how much time will a pilot expect to spend circling the field waiting to land ? [By permission from Sasieni, Yaspan and Friedman (1959)]

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Here we have an M/D/1 queue with $\lambda = 6$ per hour, and $\mu = 12$ per hour, so that $\rho = \lambda/\mu = 0.5$. Therefore,

$$L_q = \rho^2/\{2(1-\rho)\} = 1/4.$$

Hence, $W_q = L_q/\lambda = (1/24)$ hour $= 2{\cdot}5$ minutes.

Ex. 4. In a certain bank, customers arrive according to a Poisson distribution with a mean of 10 per hour. From observations on the teller's performance, the mean service time is estimated to be 4 minutes, with a variance of 8 (minutes)^2. It is felt that the Erlang would be a reasonable assumption for the distribution of the teller's service time. Also it is assumed that there is no limit on the number of customers joining the queue. Bank officials wish to know, on the average, how long a customer must wait before service, and how many customers are waiting for service.

Here $\lambda = 10$ per hour, $1/\mu = 4$ min. $= \dfrac{1}{15}$ hr.

Therefore, $\mu = 15$ per hour.

For an Erlang distribution, we have variance $= 1/(k\mu^2) = 8(\text{min.})^2$, where $1/\mu = 4$ min., so that $k = 2$. Hence we have an M/E_2/1 queueing system. Therefore,

$$\text{average time in queue} = W_q = \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \frac{3}{4} \times \frac{10}{15 \times 5} \text{ hr.}$$

$$= (1/10) \text{ hr.} = 6 \text{ min.}$$

and average number in queue $= L_q = \lambda W_q = 10 \times \dfrac{1}{10} = 1$ customer.

Ex. 5. In Ex. 4, show that the answers are independent of the fact that the service time distribution is Erlang.

This follows from the fact that for the system M/G/1, L_q, and therefore W_q, depend only on the mean and variance of the general service time distribution. Thus, if the mean and variance are given, as in the case in Ex. 3, the form of the distribution will not matter. [See 3.6(34).] In fact using the results for M/G/1, we have

$$L_q = \frac{\rho^2 + \lambda^2\sigma_r^2}{2(1-\rho)}$$

Here $\rho = \lambda/\mu = \dfrac{2}{3}$, $\lambda = 10$ per hour

and $\sigma_r^2 = 8(\text{min.})^2 = \dfrac{8}{60 \times 60} (\text{hr.})^2 = \dfrac{1}{450} (\text{hr.})^2$

So, $$L_q = \frac{\frac{4}{9} + 100 \times \frac{1}{450}}{2(1/3)} = 1 \text{ customer}$$

and $$W_q = L_q/\lambda = (1/10) \text{ hr.} = 6 \text{ min.}$$

3.10 The System GI/E_m/1 — The Matrix-Geometric Method

The matrix-geometric method [Neuts (1981)] provides a technique for analyzing queueing systems with fairly general assumptions. In what follows we use this method to study the system size behaviour of the system GI/E_m/1.

Starting, for simplicity, with the case $m = 2$, we consider the queueing system GI/E_2/1, i.e. a single-server system with general inter-arrival times and with each service time consisting of two exponential phases, marked in the reverse order, say (Cf. Section 3.8). Thus, each customer, entering service, will begin service at phase 2, at the end of which he goes to phase 1, the service being completed with the completion of phase 1.

We shall denote the state of the system by (n, r), which stands for 'n in system and service in phase r', $n = 1, 2, 3, \ldots$ and $r = 1, 2$. However, when $n = 0$, r has no significance, so we shall use '0' to denote the empty state of the system. Using a notation similar to that of Section 3.4, let k_r denote the probability that r phases of service are completed in an interarrival time of arbitrary duration, and let $\alpha_i = 1 - \sum_{r=0}^{i} k_r$. Then, corresponding to 3.4 (6) we have the transition probability matrix **P**, given by

(1) $\mathbf{P} =$

↗	0	(1, 2)	(1, 1)	(2, 2)	(2, 1)	(3, 2)	(3, 1)	. . .
0	α_1	k_0	k_0	0	0	0	0	. . .
(1, 2)	α_3	k_2	k_3	k_1	k_1	0	0	. . .
(1, 1)	α_2	k_1	k_2	0	k_0	0	0	. . .
(2, 2)	α_5	k_4	k_5	k_2	k_3	k_0	k_1	. . .
(2, 1)	α_4	k_3	k_4	k_1	k_2	0	k_0	. . .
(3, 2)	α_7	k_6	k_7	k_4	k_5	k_2	k_3	. . .
(3, 1)	α_6	k_5	k_6	k_3	k_4	k_1	k_2	. . .
.	.	.	.	.	.	.	.	. . .
.	.	.	.	.	.	.	.	. . .
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By an extension of the notation, wherein we use

(2) $$\mathbf{K}_r = \begin{pmatrix} k_{2r} & k_{2r+1} \\ k_{2r-1} & k_{2r} \end{pmatrix}$$

with $k_{-1} = 0$ in $\mathbf{K}_0$, (1) can be simplified to

(3) $$\mathbf{P} = \begin{pmatrix} \alpha_1 & \mathbf{k}_{01} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots \\ \mathbf{k}_{10} & \mathbf{K}_1 & \mathbf{K}_0 & \mathbf{O} & \mathbf{O} & \cdots \\ \mathbf{k}_{20} & \mathbf{K}_2 & \mathbf{K}_1 & \mathbf{K}_0 & \mathbf{O} & \cdots \\ \mathbf{k}_{30} & \mathbf{K}_3 & \mathbf{K}_2 & \mathbf{K}_1 & \mathbf{K}_0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

Evidently, in (3), we have

(4) $\mathbf{k}_{01} = (k_0\ k_1)$, a row vector

and

(5) $\mathbf{k}_{i0} = \begin{pmatrix} \alpha_{2i+1} \\ \alpha_{2i} \end{pmatrix}$, a column vector, $i = 1, 2, \ldots$

To find the steady-state solution for the GI/E_2/1 system, we have to determine a matrix

(6) $$\mathbf{p} = (p_0\ \mathbf{p}_1\ \mathbf{p}_2\ \ldots)$$

which satisfies the equations

(7) $$\mathbf{pP} = \mathbf{P}$$

and

(8) $$p_0 + \sum_{n=1}^{\infty} \mathbf{p}_n \mathbf{e} = 1$$

where p_0, a scalar, is the probability for empty state of the system, the $\mathbf{p}_n$ are row vectors

(9) $$\mathbf{p}_n = (p_{n2}\ p_{n1}),\ n = 1, 2, \ldots$$

with

(10) $p_{nr} = P(n$ in system and service in phase $r)$, $r = 1, 2$

and $\mathbf{e}$ is the column vector

(11) $$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now, Eq. (7), on using (3) and (6), implies

(12) $$p_0 = p_0\alpha_1 + \sum_{i=1}^{\infty} \mathbf{p}_i \mathbf{k}_{i0}$$

(13) $$\mathbf{p}_1 = p_0\mathbf{k}_{01} + \sum_{i=1}^{\infty} \mathbf{p}_i \mathbf{K}_i$$

(14) $$\mathbf{p}_j = \sum_{i=j-1}^{\infty} \mathbf{p}_i \mathbf{K}_{i-j+1},\ j = 2, 3, \ldots$$

Analogously with 3.4(9), we take a trial solution

(15) $$\mathbf{p}_j = \mathbf{AB}^j,\ j = 1, 2, \ldots$$

where A is a 1×2 vector and B a 2×2 matrix. Note that the scalar p_0 being a degenerate case has to be treated separately.

Substituting from (15) in (14), we have

(16) $$\mathbf{AB}^j = \sum_{i=j-1}^{\infty} \mathbf{AB}^i \mathbf{K}_{i-j+1}$$

$$= \mathbf{AB}^{j-1} \sum_{r=0}^{\infty} \mathbf{B}^r \mathbf{K}_r$$

Thus, Eq. (15) will satisfy (14) if

(17) $$\sum_{r=0}^{\infty} \mathbf{B}^r \mathbf{K}_r = \mathbf{B}$$

Again, substitution from (15) in (13) yields

(18) $$\mathbf{AB} = p_0\mathbf{k}_{01} + \sum_{i=1}^{\infty} \mathbf{AB}^i \mathbf{K}_i$$

$$= p_0\mathbf{k}_{01} + \mathbf{A}(\mathbf{B} - \mathbf{K}_0)$$

using (17). Thus, (15) with $\mathbf{B}$ given by (17), satisfies (13) if

(19) $$p_0\mathbf{k}_{01} = \mathbf{AK}_0$$

Finally, substitution from (15) in (12) gives

(20) $$p_0(1 - \alpha_1) = \sum_{i=1}^{\infty} \mathbf{AB}^i \mathbf{k}_{i0}$$

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Now, from (5), using the definition of α_l, we have

$$\mathbf{k}_{l0} = \begin{pmatrix} 1-\sum_0^{l+1} k_r \\ \\ 1-\sum_0^{l} k_r \end{pmatrix} \tag{21}$$

$$= \mathbf{e} - \sum_{r=0}^{l} \mathbf{K}_r\mathbf{e}$$

$$= \sum_{r=l+1}^{\infty} \mathbf{K}_r\mathbf{e}$$

the last simplification following from the fact that

$$\sum_{r=0}^{\infty} \mathbf{K}_r\mathbf{e} = \mathbf{e} \tag{22}$$

which follows from (2), since $\sum_{r=0}^{\infty} k_r = 1$

Thus (20) simplifies to

$$p_0(1-\alpha_1) = \sum_{i=1}^{\infty} \sum_{r=i+1}^{\infty} \mathbf{A}\mathbf{B}^i\mathbf{K}_r\mathbf{e} \tag{23}$$

$$= \sum_{r=2}^{\infty} \sum_{i=1}^{r-1} \mathbf{A}\mathbf{B}^i\mathbf{K}_r\mathbf{e}, \text{ changing the order of summation}$$

$$= \sum_{r=1}^{\infty} \mathbf{A}(\mathbf{I}-\mathbf{B})^{-1}(\mathbf{B}-\mathbf{B}^r)\mathbf{K}_r\mathbf{e}, \text{ adding the zero term corresponding to } r=1$$

$$= \mathbf{A}(\mathbf{I}-\mathbf{B})^{-1}\left(\mathbf{B}\sum_1^{\infty}\mathbf{K}_r\mathbf{e} - \sum_1^{\infty}\mathbf{B}^r\mathbf{K}_r\mathbf{e}\right)$$

$$= \mathbf{A}(\mathbf{I}-\mathbf{B})^{-1}(\mathbf{B}(\mathbf{e}-\mathbf{K}_0\mathbf{e}) - (\mathbf{B}-\mathbf{K}_0)\mathbf{e}),$$

$$= \mathbf{A}(\mathbf{I}-\mathbf{B})^{-1}(\mathbf{I}-\mathbf{B})\mathbf{K}_0\mathbf{e}$$

$$= \mathbf{A}\mathbf{K}_0\mathbf{e}$$

$$= p_0\mathbf{k}_{01}\mathbf{e}, \text{ using (19)}$$

$$= p_0(k_0+k_1), \text{ using (4)}$$

which is seen to be true from the definition of α_1. Thus, (15) satisfies (12) also, if we use (17) and (19).

The foregoing shows that (15) satisfies the transition equations (12) through (14), if $\mathbf{B}$ is a root of the matrix equation

$$\sum_{r=0}^{\infty} \mathbf{Z}^r\mathbf{K}_r = \mathbf{Z} \tag{24}$$

provided further that (19) holds.

Now, (24) is the matrix equation corresponding to the well-known algebraic equation 3.4(14). In fact, analogously to Section 3.4, $\mathbf{B}$ is the minimal non-negative solution of (24) and is unique for $\rho < 1$ (Cf. Gross and Harris (1985), pp. 321-322), wherein a method of computation for B, by successive approximations is given. Further, we have to find $\mathbf{A}$ and p_0. To do this, we can use (19), and the normalizing condition that the total probability is 1, which in matrix notation can be written as

$$p_0 + \sum_{j=1}^{\infty} \mathbf{p}_j\mathbf{e} = 1 \tag{25}$$

or, using (15),

$$1-p_0 = \sum_{j=1}^{\infty} \mathbf{A}\mathbf{B}^j\mathbf{e} \tag{26}$$

$$= \mathbf{A}(\mathbf{I}-\mathbf{B})^{-1}\mathbf{B}\mathbf{e}$$

From (19),

$$\mathbf{A} = p_0\mathbf{k}_{01}\mathbf{K}_0^{-1} \tag{27}$$

Eliminating $\mathbf{A}$ between (26) and (27) and solving for p_0, we have

$$p_0 = [1+\mathbf{k}_{01}\mathbf{K}_0^{-1}(\mathbf{I}-\mathbf{B})^{-1}\mathbf{B}\mathbf{e}]^{-1} \tag{28}$$

Using the value of p_0 in (27) we obtain $\mathbf{A}$. Thus the steady-state probabilities p_0, $\mathbf{p}_j$, $j = 1, 2, \ldots$ are all evaluated. This completes the solution for GI/E_2/1.

For the system GI/E_m/1, the states of the system are denoted by (n, r) with $n = 1, 2, \ldots$ as before, but with $r = 1, 2, \ldots, m$. And, as before, we have the empty state $n = 0$. In this case a service starts at phase m and proceeds through phases m-1, m-2, ..., 1. The column headings in the matrix P in (1) will now read 0, $(1, m)$, $(1, m\text{-}1)$, ..., $(1, 1)$, $(2, m)$, $(2, m\text{-}1)$, ..., $(2, 1)$, $(3, m)$, ... The rest of the analysis is similar to the foregoing theory for the case $m = 2$. For example, now we shall use the matrix $\mathbf{K}_r$ given by

$$\mathbf{K}_r = \begin{pmatrix} k_{mr} & k_{mr+1} & \cdots & k_{mr+m-1} \\ k_{mr-1} & k_{mr} & \cdots & k_{mr+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ k_{mr-m+1} & k_{mr-m+2} & \cdots & k_{mr} \end{pmatrix} \tag{29}$$

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Further, we shall have

(30) $$\mathbf{k}_{01} = (k_0\ k_1 \ldots k_{m-1})$$

(31) $$\mathbf{k}_{j0} = \begin{pmatrix} \alpha_{mi+m-1} \\ \alpha_{mi+m-2} \\ \cdots \\ \alpha_{mi} \end{pmatrix}$$

and the (1st row, 1st column) element of the matrix $\mathbf{P}$ in (3) will be α_{m-1} instead of α_i. We now have to evaluate p_0, $\mathbf{p}_j$ $(j = 1, 2, \ldots)$ with p_0 a scalar as before and the $\mathbf{p}_j$ each a vector $(1\times m)$. Proceeding as for the case $m = 2$, we shall see that the solution is still given by (15), but with $\mathbf{A}$ a $(1\times m)$ vector and $\mathbf{B}$ an $(m\times m)$ matrix, which is still the minimal non-negative solution of (24), where $\mathbf{Z}$ is now $(m\times m)$ and not (2×2) as before. Thus, formally, the results will remain the same as before with the dimensions of the matrices and vectors changed. Though $\mathbf{A}$ and p_0 will still be given by (27), (28), respectively, $\mathbf{I}$ will be the identity matrix of order m and $\mathbf{e}$ a column vector consisting of m 1's.

Mean Number in System, i.e., L, is obviously given by

(32) $$\begin{aligned} L &= \sum_{j=1}^{\infty} j\mathbf{p}_j\mathbf{e} \\ &= \sum_{1}^{\infty} j\mathbf{A}\mathbf{B}^j\mathbf{e} \\ &= \mathbf{A}\sum_{1}^{\infty} j\mathbf{B}^{j-1}\mathbf{B}\mathbf{e} \\ &= \mathbf{A}(\mathbf{I}+2\mathbf{B}+3\mathbf{B}^2+\ldots)\mathbf{B}\mathbf{e} \\ &= \mathbf{A}(\mathbf{I}-\mathbf{B})^{-2}\mathbf{B}\mathbf{e} \end{aligned}$$

Also, the second moment, about the origin, of the system size distribution is

(33) $$\begin{aligned} E[X^2] &= \sum_{j=1}^{\infty} j^2\mathbf{A}\mathbf{B}^j\mathbf{e} \\ &= \mathbf{A}[(\mathbf{I}-\mathbf{B})^{-2}\mathbf{B}+2(\mathbf{I}-\mathbf{B})^{-2}\mathbf{B}^2]\mathbf{e} \end{aligned}$$

as can easily be seen by summing up the series $\sum_{1}^{\infty} j^2\mathbf{B}^j$. Hence we obtain the variance of the number in the system as

(34) $$\mathrm{Var}[X] = E[X^2]-L^2$$

where L is given by (32).

The Case of GI/M/1

It is interesting as well as instructive to obtain results for GI/M/1 as a special case from the above analysis by setting $m = 1$. Evidently for the case $m = 1$ we shall have $\mathbf{I} = \mathbf{e} = 1$, $\mathbf{K}_0 = \mathbf{k}_{01} = k_0$ and $\mathbf{B} = b$, the root of $\sum_{0}^{\infty} b^r k_r = b$ lying between 0 and 1. Thus, (28) will reduce to

(35) $$p_0 = [1+(1-b)^{-1}b]^{-1} = 1-b$$

as is well-known. Similarly (27) will become

(36) $$\mathbf{A} = p_0 = 1-b$$

and thus (15) will become

(37) $$p_j = (1-b)b^j$$

Note. A further generalization of the system GI/E_m/1 discussed in this section is to GI/PH/1, where 'PH' stands for a 'phase-type distribution'. Interested readers may refer to Neuts (1981).

PROBLEM SET 3

1. Repairing a certain type of machine which breaks down in a given factory consists of five basic steps that must be performed sequentially. The time taken to perform each of the five steps is found to have an exponential distribution with mean 5 minutes and is independent of the other steps. If these machines break down in a Poisson fashion at an average rate of two per hour, and if there is only one repairman, what is the average idle time for each machine that has broken down ?

[1 *hr.* 40 *min.*]

2. In a car manufacturing plant, a loading crane takes exactly 10 minutes to load a car into a wagon and again come back to the position to load another car. If the arrival of cars is a Poisson stream at an average of one every 20 minutes, calculate the average waiting time of a car.

[5 *minutes*]

3. The queueing system of machines needing repair at a repair shop behaves like M/E_2/1 with $\lambda = 6/5$ per hour, and $\mu = 3/2$ per hour. Find the average duration of time a machine has to wait for repair and also the average time spent in the shop.

[2 *hours*; 2 *hours* 40 *minutes*]

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4. A warehouse in a small state receives orders for a certain item and sends them by a truck as soon as possible to the customer. The orders arrive in a Poisson fashion at a mean rate of 0·9 per day. Only one item at a time can be shipped by truck from the warehouse, which is located in the central part of the state. Because the customers are located in various places in the state, the distribution of service time in days is one with probability density $4te^{-2t}$. What is the expected delay between the arrival of an order and the arrival of the item to the customer ? Service time here implies the time the truck takes to load, get to the customer, unload and return to the warehouse. Loading and unloading times are small compared with travel time. [By permission from Sasieni, Yaspan and Friedman (1959)] [$7\frac{1}{2}$ *days*]

5. A colliery working one shift per day uses a large number of locomotives which break down at random intervals; on an average one fails per 8 hour shift. The fitter carries out a standard maintenance schedule on each faulty loco. Each of the five main parts of this schedule takes an average 1/2 hour, but the time varies widely. How much time will the fitter have available for other tasks per day and what is the average time a loco is out of service ?

[5·5 *hours*; 3·18 *hours*]

6. In an M/G/1 queueing system, let Y_n be the number of customers left behind in the system by the n-th departing customer, and X_n the number of arrivals during the service time of the n-th customer. Show that

$$Y_{n+1} = \max(Y_n - 1, 0) + X_{n+1} = Y_n - U(Y_n) + X_{n+1}$$

where $U(x)$ is Heaviside's unit function, given by

$$U(x) = 1 \quad (x > 0)$$
$$= 0 \quad (x \leqslant 0)$$

Using the fact that in the steady-state the expectation $E[Y_{n+1}] = E[Y_n]$ and $E[Y_{n+1}^2] = E[Y_n^2]$, deduce that the steady-state expectation $E[Y]$ is given by

$$E[Y] = \rho + \frac{\rho^2(1+C_v^2)}{2(1-\rho)}$$

where C_v is the coefficient of variation of service times, and ρ is the traffic intensity.

[*Hint.* Equating expectations of Y_{n+1} and Y_n gives $E[U(Y_n)] = E[X_{n+1}]$.

Now,

$$E[X_{n+1}] = \int_0^\infty \sum_{r=1}^\infty r.[(\lambda v)^r e^{-\lambda v}/r\,!]\, dB(v) = \rho$$

where $B(v)$ is the service time distribution function.

Similarly,

$$E[X_{n+1}^2] = \lambda^2 E[v^2] + \rho = \lambda^2[(1/\mu^2) + \sigma_v^2] + \rho$$
$$= \rho^2[1 + C_v^2] + \rho$$

Note that $xU(x) = x$, so that $[U(x)]^2 = U(x)$. Now the result follows by equating the expectations of Y_{n+1}^2 and Y_n^2. Assume that Y_n and X_{n+1} are independent.]

7. In M/G/1, show that the busy period distribution function $G(x)$ is given by

$$G(x) = \int_0^x \sum_{n=0}^\infty \{(\lambda v)^n e^{-\lambda v}/n\,!\}\, G_n(x-v)\, dB(v)$$

where $B(v)$ is the service time distribution and $G_n(x)$ denotes the n-fold convolution of $G(x)$ with itself. Hence deduce the functional equation

$$g(s) = b(s + \lambda - \lambda g(s))$$

where

$$g(s) = \int_0^\infty e^{-sx}\, dG(x)$$

is the Laplace-Stieltjes transform (L.S.T.) of $G(x)$, and $b(s)$ is the L.S.T. of $B(v)$. Verify the expression for $g(s)$ obtained in 2.3. (9) for the M/M/1 system.

[*Hint.* Regard the busy period as made up of the service time of the first customer plus a busy period initiated by n customers in the system, provided that there are n arrivals during the service time. Now use the argument of Section 2.3 for the initial busy period.

For M/M/1, $b(s) = \mu/(\mu + s)$; thus we have a quadratic equation in $g(s)$, giving $g(s) = [(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}]/2\lambda$. Note that minus sign is taken before the radical, since we shall have $g(\infty) = 0$.]

8. For the system GI/M/2, show that the steady-state probabilities π_j are given by

$$\pi_j = Cb^j \quad (j \geqslant 1)$$

where b is the root, in $0 < \theta < 1$, of the equation

$$\theta = a(2\mu - 2\mu\theta)$$

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and

$$C = (1-b)(1-2b)\tilde{a}(\mu)/[b(1-b-\tilde{a}(\mu))]$$

$$\pi_0 = (1-b)(1-2\tilde{a}(\mu))/[1-b-\tilde{a}(\mu)]$$

and $\tilde{a}(s)$ is the L.S.T. of the arrival distribution $A(u)$.

[*Hint.* For evaluation of C and π_0, proceed as follows. Considering one-step transitions, we have

(1) $$\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + (1-\pi_0-\pi_1)p_{q0}$$

where

(2) $$p_{00} = \int_0^\infty (1-e^{-\mu u})dA(u) = 1-\tilde{a}(\mu)$$

(3) $$p_{10} = \int_0^\infty (1-e^{-\mu u})^2 dA(u) = 1-2\tilde{a}(\mu)+\tilde{a}(2\mu)$$

and p_{q0} is the probability that a customer, who arrives to find two or more units in the system and, therefore, has to wait, is followed by the next arrival who finds the system empty. Let the interval between these two arrivals be u. Then, if the queueing time of the first customer is w, we should require both the servers to complete service within a duration $u-w$. Also w follows the distribution given by 3.5(9). Therefore, we have

$$p_{q0} = \int_0^\infty \int_0^u 2(1-b)\mu e^{-2(1-b)\mu w}[1-e^{-\mu(u-w)}]^2 dw dA(u)$$

which, on using the relation $\tilde{a}(2\mu-2\mu b) = b$, simplifies to

(4) $$p_{q0} = \frac{1-b}{1-2b}\left[2-4\tilde{a}(\mu)-\frac{1-2b}{b}\tilde{a}(2\mu)\right]$$

The results follow on using (1) through (4) and the facts that $\pi_j = Cb^j$ $(j \geqslant 1)$ and $\sum_{j=0}^{\infty} \pi_j = 1$.]

9. For the system GI/E$_m$/1, show that k_r as defined in Section 3.10 is given by

$$k_r = (-1)^r\{(m\mu)^r/r!\}\tilde{a}^{(r)}(m\mu)$$

where μ is the mean service rate, and

$$\tilde{a}^{(r)}(m\mu) = \frac{d^r}{ds^r}\tilde{a}(s)\bigg|_{s=m\mu}$$

is the r-th derivative of the Laplace transform, $\tilde{a}(s)$, of the inter-arrival time density function, evaluated at $s = m\mu$. Deduce the expressions for k_r for the systems :

(i) E$_l$/E$_m$/1, (ii) D/E$_m$/1.

[By definition,

$$k_r = \int_0^\infty \{(m\mu u)^r e^{-m\mu u}/r!\}a(u)du$$

whence the result follows on using the L.T. relation (9) of Appendix E.

(i) For E$_l$/E$_m$/1, $\tilde{a}(s) = \left(\frac{l\lambda}{l\lambda+s}\right)^l$, whence on performing the successive differentiation and simplifying, we have

$$k_r = \binom{l+r-1}{r} p^l q^r$$

with

$$p = l\lambda/(l\lambda+m\mu), \quad q = 1-p$$

which is a negative-binomial distribution.

(ii) For D/E$_m$/1, $\tilde{a}(s) = e^{-s/\lambda}$. Proceeding as before, we have

$$k_r = (m\mu/\lambda)^r e^{-m\mu/\lambda}/r!$$

which is a Poisson distribution.]

10. For Ex. 2 of this chapter (p. 67), evaluate the average number of patients waiting in the doctor's office. Also obtain the average number waiting, when :

(i) the inter-arrival times have a 2-Erlang distribution with the same mean as before,

(ii) the inter-arrival times are constant and equal to 20 minutes. Note that this is the case when the doctor makes firm appointments and the patients adhere to the same.

[2; 1·6636; 0·5362]

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CHAPTER 4

BULK, TANDEM AND PRIORITY QUEUES

4.1 Introduction

In this chapter we discuss queueing systems in which either the arrivals take place in groups or the service is rendered in groups. The size of the groups will be regarded as a random variable given by a probability distribution. The queueing systems described in earlier chapters with group size always equal to one become particular cases of the models discussed here. Also included in the chapter are queues in tandem and priority queues.

4.2 The Queueing System $M^X/M/1$

This is a single-server bulk-arrival queueing system where the times between consecutive arrival epochs and the service times are both exponentially distributed with means $1/\lambda$ and $1/\mu$, and, at each arrival instant, customers arrive in groups of variable size X, given by the probability distribution

$$P(X = r) = a_r, \quad r = 1, 2, 3, \ldots \tag{1}$$

It is easy to see that in the steady-state, corresponding to Eq. 1.5(7) and (8), we shall have

$$0 = -(\lambda+\mu)P_n + \lambda \sum_{r=1}^{n} P_{n-r}a_r + \mu P_{n+1}, \quad n = 1, 2, \ldots \tag{2}$$

$$0 = -\lambda P_0 + \mu P_1 \tag{3}$$

Define the following generating function :

$$P(z) = \sum_{n=0}^{\infty} P_n z^n, \quad |z| \leqslant 1 \tag{4}$$

From Eq. (2) and (3) we have, as usual

$$0 = -(\lambda+\mu)P(z) + \mu P_0 + \mu \sum_{n=0}^{\infty} P_{n+1}z^n + \lambda \sum_{n=1}^{\infty} \sum_{r=1}^{n} P_{n-r}a_r z^n \tag{5}$$

The double summation in the last term, on changing the order of summation, gives

$$\sum_{r=1}^{\infty} \sum_{n=r}^{\infty} (P_{n-r}z^{n-r})a_r z^r = P(z)A(z) \tag{6}$$

where

$$A(z) = \sum_{r=1}^{\infty} a_r z^r \tag{7}$$

is the generating function of the group size. Thus (5) finally yields

$$P(z) = \frac{\mu(1-z)P_0}{\lambda z A(z) - (\lambda+\mu)z + \mu} \tag{8}$$

Now, since $P(1-0) = 1$, on using l'Hôpital's rule we have from (8)

$$P_0 = 1 - (\lambda\bar{a}/\mu) \tag{9}$$

where $\bar{a} = \Sigma\, r a_r$ is the mean group size, and we use the following obvious relations

$$A(1) = 1 \tag{10}$$

$$A'(1) = \frac{d}{dz} A(z)\Big|_{z=1} = \bar{a} \tag{11}$$

It is clear that the traffic intensity for this system is given by

$$\rho = \lambda\bar{a}/\mu \tag{12}$$

So (9) shows that we have the usual relation

$$P_0 = 1 - \rho \tag{13}$$

Thus, for the steady state to exist, $\rho < 1$.

Substituting from (9) in (8) and then evaluating the derivative at $z = 1-0$ with the help of l'Hopital's rule, we see that the mean number in the system is given by

$$L = P'(1-0) = \frac{\rho}{1-\rho}\{1 + A''(1)/2\bar{a}\} \tag{14}$$

where

$$A''(1) = \frac{d^2}{dz^2} A(z)\Big|_{z=1}$$

$$= \sum_{r=1}^{\infty} (r^2 - r)a_r \tag{15}$$

$$= \sigma_a^2 + \bar{a}^2 - \bar{a}$$

σ_a^2 being the variance of the group size.

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From (13) it follows that $E[N_{serv}] = \rho$, and hence

$$L_q = L - \rho \tag{16}$$

In the case of single arrivals we have

$$A(z) = z, \quad \bar{a} = 1, \quad A''(1) = 0$$

and, therefore, (14) reduces to

$$L = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho} \tag{17}$$

which is a well-known result for M/M/1.

Waiting Time

We now consider the distribution of the *time spent in system* by an arbitrary customer. It may be noted that in the system $M^x/M/1$ arrivals occur in groups of variable size. We shall assume that within an arriving group the customers are served in random order. Let us first take the first-served customer of an arriving group. If on his arrival he finds n customers in the system, he will stay in the system until the completion of $(n+1)$ services. Therefore, the waiting time (in the system) of the first-served customer of a group has the density function $f_1(w)$, given by

$$f_1(w)dw = \sum_{n=0}^{\infty} P_n \cdot \frac{(\mu w)^n e^{-\mu w}}{n\,!} \cdot \mu dw \tag{18}$$

Hence, the Laplace transform of $f_1(w)$ is given by

$$\tilde{f}_1(s) = \int_0^{\infty} e^{-sw} f_1(w) dw \tag{19}$$

$$= \sum_{n=0}^{\infty} P_n \left(\frac{\mu}{\mu+s} \right)^{n+1} = \frac{\mu}{\mu+s} P\left(\frac{\mu}{\mu+s} \right)$$

where $P(z)$ is defined by (4) and is given by (8), (9).

Now, if a customer is served as the j-th customer in his group, he will have to wait for an additional time during which $(j-1)$ services are completed. Also, it can be shown that in random selection for service the probability of a customer being served j-th is

$$p_j = (1 - \sum_{r=1}^{j-1} a_r)/\bar{a} \tag{20}$$

Therefore, as before, the L.T., $\tilde{f}_2(s)$, of the p.d.f. of the additional waiting time of an arbitrary customer over the first customer is

$$\tilde{f}_2(s) = \sum_{j=1}^{\infty} p_j \left(\frac{\mu}{\mu+s} \right)^{j-1} \tag{21}$$

On substituting for p_j from (20) in (21), we have

$$\tilde{f}_2(s) = \frac{1}{\bar{a}} \sum_{j=1}^{\infty} \left(1 - \sum_{r=1}^{j-1} a_r\right) \left(\frac{\mu}{\mu+s} \right)^{j-1} \tag{22}$$

$$= \frac{1}{\bar{a}} \left[\frac{1}{1 - \frac{\mu}{\mu+s}} - \sum_{j=2}^{\infty} \sum_{r=1}^{j-1} a_r \left(\frac{\mu}{\mu+s} \right)^{j-1} \right]$$

$$= \frac{1}{\bar{a}} \left[\frac{\mu+s}{s} - \sum_{r=1}^{\infty} \sum_{j=r+1}^{\infty} a_r \left(\frac{\mu}{\mu+s} \right)^{r} \left(\frac{\mu}{\mu+s} \right)^{j-r-1} \right]$$

$$= \frac{1}{\bar{a}} \cdot \frac{\mu+s}{s} \left[1 - A\left(\frac{\mu}{\mu+s} \right) \right]$$

Hence, using the convolution theorem of Laplace transforms, we have for the L.T. of the p.d.f. of the waiting time in the system for an arbitrary customer :

$$\tilde{f}(s) = \tilde{f}_1(s)\tilde{f}_2(s) \tag{23}$$

$$= \frac{\mu}{\bar{a}s} \left[1 - A\left(\frac{\mu}{\mu+s} \right) \right]$$

Differentiating $\tilde{f}(s)$ w.r.t. s and taking limits as $s \to 0$ with the help of l'Hôpital's rule, we have for the mean waiting time in the system

$$W = -\tilde{f}'(0) = \frac{1}{\mu} \left[L + 1 + \left(\frac{1}{2\bar{a}} \right) A''(1) \right] \tag{24}$$

If we eliminate $A''(1)$ between (14) and (24), we obtain

$$L = \lambda \bar{a} W \tag{25}$$

which shows that the well-known Little's formula holds in this case also. Note that here the mean number of customers arriving per unit time is $\lambda\bar{a}$.

Queueing Time

Proceeding as above, we can show that the L.T. of the p.d.f. of the time spent in the queue by an arbitrary customer is

$$\tilde{f}_q(s) = \tilde{f}_{q1}(s)\tilde{f}_2(s) \tag{26}$$

where $\tilde{f}_2(s)$ is as given in (21), (22), and $\tilde{f}_{q1}(s)$ is the L.T. of p.d.f. of the queueing time of the first customer of a group, given by

$$\tilde{f}_{q1}(s) = \sum_{n=0}^{\infty} P_n \left(\frac{\mu}{\mu+s} \right)^n = P\left(\frac{\mu}{\mu+s} \right) \tag{27}$$

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Also, we have for the mean number in the queue

$$L_q = \sum_{n=1}^{\infty} (n-1)P_n = \sum_{n=0}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n \tag{28}$$

$$= L-(1-P_0) = L-\rho$$

And the mean queueing time is given by

$$W_q = -\tilde{f}_q'(0) \tag{29}$$

From Eq. (26) through (29) and (14), we can show that

$$L_q = \lambda\bar{a}W_q \tag{30}$$

which verifies the other relation in Little's formula.

4.3 The Queueing System M/M^b/1

This is the single-server bulk service queueing system with Poisson arrivals and exponential service times, in which whenever the server finds a number of customers waiting, he begins service on b of them or the whole queue, whichever is less. Let P_n be the steady-state probability that n units are waiting in the queue ($n = 0, 1, 2, \ldots$) and the server is busy, and let P_{00} be the probability that the server is idle (the service station is empty).

The steady-state transition equations are

$$(\lambda+\mu)P_n = \lambda P_{n-1}+\mu P_{n+b} \quad (n \geqslant 1) \tag{1}$$

$$(\lambda+\mu)P_0 = \lambda P_{00}+\mu \sum_{j=1}^{b} P_j \tag{2}$$

$$\lambda P_{00} = \mu P_0 \tag{3}$$

Let us introduce the generating function

$$P(z) = \sum_{n=0}^{\infty} P_n z^n, \quad |z| \leqslant 1 \tag{4}$$

From (1) and (2), using (3) we have

$$(\lambda+\mu-\lambda z-\mu z^{-b})P(z) = \mu \sum_{n=0}^{b} P_n(1-z^{n-b}) \tag{5}$$

whence

$$P(z) = \frac{\mu \sum_{n=0}^{b-1} P_n(z^n-z^b)}{\lambda z^{b+1}-(\lambda+\mu)z^b+\mu} \tag{6}$$

Now, let us take

$$f(z) = -(\lambda+\mu)z^b, \quad g(z) = \lambda z^{b+1}+\mu \tag{7}$$

and take a closed contour C given by $|z| = 1+\delta$, where δ is small. Then, we have, on C,

$$|f(z)| = |(\lambda+\mu)z^b| = |\lambda+\mu|\cdot|z|^b = (\lambda+\mu)(1+\delta)^b \\ = (\lambda+\mu)(1+b\delta)+O(\delta^2) \tag{8}$$

and,

$$|g(z)| = |\lambda z^{b+1}+\mu| \leqslant |\lambda|\cdot|z|^{b+1}+|\mu| \\ = \lambda\{1+(b+1)\delta\}+\mu+O(\delta^2) \\ = \lambda+\mu+\lambda(b+1)\delta+O(\delta^2) \tag{9}$$

Therefore, we can write

$$|g(z)| < |f(z)| \quad \text{on } C \tag{10}$$

if

$$\lambda\delta < \mu b\delta \tag{11}$$

i.e., if

$$\rho = \frac{\lambda}{b\mu} < 1 \tag{12}$$

which is the condition for the steady state to exist. When this condition is satisfied, using Rouché's theorem we can write that the denominator of the right-hand member of (6), which is $f(z)+g(z)$, has the same number of zeros inside the contour C as $f(z)$. But $f(z)$ has b zeros inside C; so the denominator $f(z)+g(z)$ also has b zeros inside C, one of which is $z = 1$ and therefore the remaining $b-1$ zeros are inside the unit circle $|z| = 1$. Hence the denominator has only one zero outside $|z| = 1$; let us call it z_0.

Now, corresponding to each zero z_i of an expression, the expression should have a factor $(z-z_i)$. Therefore b of the factors of the denominator on the right-hand side of (6) should cancel with the b similar factors of the numerator. Also the numerator is of degree b and the denominator is of degree $b+1$. So, on cancelling the like factors, (6) can be written

$$P(z) = \frac{A}{z_0-z} = A \sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}} \tag{13}$$

where A is a constant. The expansion given in (13) is valid since clearly $|z/z_0| < 1$. Noting that the coefficient of z^n in $P(z)$ is P_n, we have, on setting $z = 0$,

$$A = P_0 z_0 \tag{14}$$

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Thus, (13) becomes

$$P(z) = P_0 \sum_{n=0}^{\infty} (z/z_0)^n \tag{15}$$

Therefore, we have

$$P_n = P_0 z_0^{-n}, \quad n = 0, 1, 2, \ldots \tag{16}$$

Also, from (3),

$$P_0 = (\lambda/\mu)P_{00} = b\rho P_{00} \tag{17}$$

Thus, (16) yields

$$P_n = b\rho P_{00} z_0^{-n}, \quad n = 0, 1, 2, \ldots \tag{18}$$

Finally, P_{00} can be evaluated from the normalizing condition

$$P_{00} + \sum_{n=0}^{\infty} P_n = 1 \tag{19}$$

which, after simplification, gives

$$P_{00} = \frac{z_0 - 1}{(1+b\rho)z_0 - 1} \tag{20}$$

This completes the steady-state probability distribution of the number in the queue.

The mean number in the queue is

$$L_q = \sum_{n=1}^{\infty} nP_n = P_0 \left(\frac{1}{z_0} + \frac{2}{z_0^2} + \frac{3}{z_0^3} + \ldots \right) \tag{21}$$

$$= \frac{P_0 z_0}{(z_0 - 1)^2}$$

Using (17) and (20) we have

$$L_q = \frac{b\rho z_0}{(z_0 - 1)\{(1+b\rho)z_0 - 1\}} \tag{22}$$

Note that (21) could also be obtained by differentiating (13) at $z = 1$.

It may be observed that z_0 is the root, outside the unit circle $|z| = 1$, of the equation

$$\lambda z^{b+1} - (\lambda+\mu)z^b + \mu = 0 \tag{23}$$

which can be written

$$\frac{1}{z} = \frac{\lambda}{\lambda + \mu - \mu z^{-b}} \tag{24}$$

Setting $z = \frac{1}{x}$, we see that $1/z_0$ is the root, within the unit circle, of the equation

$$x = \frac{\lambda}{\lambda + \mu - \mu x^b} = \tilde{a}(\mu - \mu x^b) \tag{25}$$

where $\tilde{a}(\cdot)$ is the L.T. of the inter-arrival time density function, since we have

$$\tilde{a}(s) = \lambda/(\lambda + s) \tag{26}$$

Particular case : M/M/1. Setting $b = 1$, it is clear that $z_0 = \mu/\lambda = 1/\rho$ and we can verify the various results of M/M/1 from the foregoing. In particular, from (20) and (18),

$$P_{00} = 1 - \rho \tag{27}$$

$$P_n = P(n+1 \text{ in system}) = (1-\rho)\rho^{n+1}, \quad n = 0, 1, 2, \ldots \tag{28}$$

and, from (22),

$$L_q = \rho^2/(1-\rho) \tag{29}$$

Queueing Time

Since the waiting customers are taken in for service in batches of b customers or the whole queue, whichever is less, a customer who finds $(ib+j)$ customers waiting in the queue $(0 \leqslant j \leqslant b-1)$ has to wait until $(i+1)$ service times are over—i for the ib customers preceded by one for the residual service of the batch being served. Therefore, the queueing time density function $f_q(w)$ is given by

$$f_q(w)dw = \sum_{i=0}^{\infty} \sum_{j=0}^{b-1} P_{ib+j} \frac{(\mu w)^i}{i!} e^{-\mu w} \mu dw \tag{30}$$

$$P(w = 0) = P_{00} \tag{31}$$

Substituting from (16) in (30), we have

$$f_q(w)dw = \sum_{i=0}^{\infty} \sum_{j=0}^{b-1} P_0 z_0^{-ib-j} \frac{(\mu w)^i}{i!} \mu e^{-\mu w} dw \tag{32}$$

$$= P_0 \sum_{j=0}^{b-1} z_0^{-j} \sum_{i=0}^{\infty} \frac{(\mu w z_0^{-b})^i}{i!} \mu e^{-\mu w} dw$$

$$= P_0 \left(\frac{1 - z_0^{-b}}{1 - z_0^{-1}} \right) \cdot \mu \exp[-\mu w(1 - z_0^{-b})]$$

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Thus, the mean queueing time is

$$W_q = \int_0^\infty w f_q(w)\,dw \tag{33}$$

$$= P_0 \cdot \frac{1-z_0^{-b}}{1-z_0^{-1}} \cdot \mu \cdot \frac{1}{\mu^2(1-z_0^{-b})^2}$$

$$= \frac{P_0}{\mu(1-z_0^{-1})(1-z_0^{-b})}$$

Now, z_0 is a root of (23); therefore, we have

$$z_0^{-b} = (\lambda+\mu-\lambda z_0)/\mu = 1+b\rho(1-z_0) \tag{34}$$

Hence, (33) simplifies to

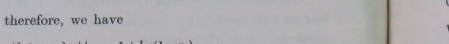
$$W_q = \frac{P_0 z_0}{\mu(z_0-1)\cdot b\rho(z_0-1)} \tag{35}$$

$$= \frac{1}{\mu b\rho} \cdot \frac{P_0 z_0}{(z_0-1)^2} = L_q/\lambda$$

using (12) and (21). Thus we see that Little's formula holds for this system.

4.4 Queues in Series (or in Tandem)

Let us consider two service facilities arranged in series as shown in the figure. Let customers arrive in a Poisson fashion with mean rate λ for



service at service station 1. After completing service at service station 1 the units join another queue for service at service station 2. Let the service times at stations 1 and 2 be both exponentially distributed with mean service rates μ_1 and μ_2 respectively. This system thus consists of two distinct queueing systems which we shall call subsystems 1 and 2. Let $P_{n_1,\,n_2}(t)$ be the probability that at time t there are n_1 units in the first subsystem (in queue or in service) and n_2 units in the second subsystem. Then, we have

$$\begin{aligned} P_{n_1,\,n_2}(t+\Delta t) = {} & P_{n_1,\,n_2}(t)\{1-(\lambda+\mu_1+\mu_2)\Delta t\} \\ & + P_{n_1-1,\,n_2}(t)\cdot\lambda\Delta t + P_{n_1+1,\,n_2-1}(t)\cdot\mu_1\Delta t \\ & + P_{n_1,\,n_2+1}(t)\cdot\mu_2\Delta t + o(\Delta t), \quad (n_1 \geqslant 1,\ n_2 \geqslant 1) \end{aligned} \tag{1}$$

whence we have, for the steady-state probabilities $P_{n_1,\,n_2}$,

$$0 = -(\lambda+\mu_1+\mu_2)P_{n_1,\,n_2} + \lambda P_{n_1-1,\,n_2} + \mu_1 P_{n_1+1,\,n_2-1} + \mu_2 P_{n_1,\,n_2+1} \tag{2}$$
$$(n_1 \geqslant 1,\ n_2 \geqslant 1)$$

Similarly, we have the following special cases of this equation :

$$0 = -(\lambda+\mu_1)P_{n_1,\,0} + \lambda P_{n_1-1,\,0} + \mu_2 P_{n_1,\,1}, \quad (n_1 \geqslant 1) \tag{3}$$

$$0 = -(\lambda+\mu_2)P_{0,\,n_2} + \mu_1 P_{1,\,n_2-1} + \mu_2 P_{0,\,n_2+1}, \quad (n_2 \geqslant 1) \tag{4}$$

$$0 = -\lambda P_{00} + \mu_2 P_{01} \tag{5}$$

We take as a trial solution

$$P_{n_1,\,n_2} = C\rho_1^{n_1}\rho_2^{n_2} \tag{6}$$

where C is a constant and $\rho_1 = \lambda/\mu_1$, $\rho_2 = \lambda/\mu_2$.

Dividing (2) throughout by λ and substituting from (6), we have

$$0 = -(1+\rho_1^{-1}+\rho_2^{-1})C\rho_1^{n_1}\rho_2^{n_2} + C\rho_1^{n_1-1}\rho_2^{n_2} + C\rho_1^{n_1}\rho_2^{n_2-1} + C\rho_1^{n_1}\rho_2^{n_2} \tag{7}$$

which is true. Hence the trial solution (6) satisfies (2). Similarly, it can also be seen to satisfy (3), (4) and (5). Also, using the normalizing condition

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} P_{n_1,\,n_2} = 1 \tag{8}$$

we have

$$C = (1-\rho_1)(1-\rho_2) \tag{9}$$

and thus we arrive at the solution

$$P_{n_1,\,n_2} = (1-\rho_1)(1-\rho_2)\rho_1^{n_1}\rho_2^{n_2} \tag{10}$$

The marginal probability of n_1 units in the first subsystem irrespective of the number in subsystem 2 is

$$P_{n_1,\,\cdot} = \sum_{n_2=0}^{\infty} P_{n_1,\,n_2} = (1-\rho_1)\rho_1^{n_1} \tag{11}$$

Similarly, the marginal probability of n_2 in subsystem 2 is

$$P_{\cdot,\,n_2} = \sum_{n_1=0}^{\infty} P_{n_1,\,n_2} = (1-\rho_2)\rho_2^{n_2} \tag{12}$$

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From (10)–(12), we observe the following :

(a) The numbers of units in the two subsystems are statistically independent.

(b) The first subsystem is an M/M/1 system with traffic intensity ρ_1.

(c) The second subsystem is also an M/M/1 system with traffic intensity ρ_2, and since the mean service rate is μ_2, the mean arrival rate should be λ.

Now, the observation (b) above is not at all surprising, since subsystem 1 could be studied separately, without caring as to what happens to its output, as M/M/1. On the other hand, (c) is noteworthy, since subsystem 2 does not experience free input and λ is given as the mean arrival rate at subsystem 1. The observation (c) implies that the inter-departure times at subsystem 1 are exponentially distributed with mean $1/\lambda$. This important property about the output of a Poisson queue will be discussed later.

From (11) and (12) we see that the mean number in the first subsystem is

$$L_1 = \rho_1/(1-\rho_1) \tag{13}$$

and the mean number in the second subsystem is

$$L_2 = \rho_2/(1-\rho_2) \tag{14}$$

Therefore, the mean number in the entire system is

$$L = L_1+L_2 = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} \tag{15}$$

Eq. (15) can alternatively be derived by noting that with probability P_{n_1, n_2} there are n_1+n_2 units in the entire system. Thus,

$$L = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (n_1+n_2) P_{n_1, n_2} \tag{16}$$

Now, using (10) and evaluating the double summation we have (15).

In fact, the observation (c) above implies that the two subsystems can be studied separately as M/M/1 systems each with mean arrival rate λ. Further, if there are any more subsystems added after the second which have exponential service times, they will all experience Poisson arrivals with mean rate λ.

Ex. 1. In an M/M/1 queue, show that the time between successive departures has the same probability distribution as the inter-arrival time distribution.

Let λ, μ be the mean arrival and service rates, and let $\rho = \lambda/\mu$.

Now, with probability $1-P_0 = \rho$, a departing customer leaves behind a non-empty queue and then the time to next departure is the same as a service time. Also, with probability $P_0 = 1-\rho$, the departing customer leaves an empty station, and in this case the time to the next departure consists of the time until the next arrival followed by a service time. Furthermore, by the Markovian property of the arrival distribution, the time to the next arrival has the same distribution as an inter-arrival time. Thus, in this second case the inter-departure time is distributed as the sum of an inter-arrival time and a service time.

Therefore, the density function $f(t)$ of the inter-departure times is given by

$$f(t) = (1-P_0)b(t)+P_0 a(t)*b(t)$$

where $a(t)$ and $b(t)$ are the density functions of the inter-arrival time and service time, respectively, and the asterisk (*) stands for convolution. Thus, we have

$$\begin{aligned} f(t) &= \frac{\lambda}{\mu} \cdot \mu e^{-\mu t} + \left(1-\frac{\lambda}{\mu}\right) \int_0^t \lambda e^{-\lambda u} \cdot \mu e^{-\mu(t-u)} du \\ &= \lambda e^{-\mu t} + \lambda(\mu-\lambda) e^{-\mu t} \int_0^t e^{(\mu-\lambda)u} du \\ &= \lambda e^{-\mu t} \left[1+(\mu-\lambda) \cdot \frac{e^{(\mu-\lambda)t}-1}{\mu-\lambda}\right] \\ &= \lambda e^{-\lambda t} \end{aligned}$$

which is the same as the inter-arrival time density function.

4.5 Non-Preemptive Priorities

The concept of priorities was introduced in Section 1.4. We shall now consider the waiting time (in queue) of customers in the case of non-preemptive priority or head-of-the-line priority rule, where a service once started is allowed to continue until completion, as explained in Section 1.4.

Let there be r priority classes, called classes $1, 2, \ldots, r$, class 1 having the highest priority and class r the lowest priority. Let there be a single server and let customers of priority class k arrive according to a Poisson distribution with parameter λ_k $(k = 1, 2, \ldots, r)$. Thus the arrivals in different classes have different Poisson distributions. The combined arrival distribution will be Poisson with parameter

$$\lambda = \sum_{k=1}^{r} \lambda_k \tag{1}$$

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Let the service time distribution for class k have a probability distribution $B_k(t)$, $(k = 1, 2, \ldots, r)$ with mean $1/\mu_k$, so that the combined service time distribution is

$$B(t) = (1/\lambda) \sum_{k=1}^{r} \lambda_k B_k(t) \tag{2}$$

We assume the queue discipline within a priority class to be first-come first-served.

The queueing time of a customer of any class will be zero if he finds the server idle and therefore no customers waiting, on his arrival. To find the mean queueing time of a customer of a priority class p, say W_p, suppose on arrival he finds the server busy and n_k units of priority class k $(k = 1, 2, \ldots, p)$ waiting in queue. Let T be the queueing time of the customer of class p under consideration. During time T, let n_k' units of class k $(k = 1, 2, \ldots, p-1)$ arrive, who will all be served before the unit being considered. Let T_0 be the time required to complete the service already in progress, T_k the time to serve the n_k units already waiting $(k = 1, 2, \ldots, r)$, and T_k' the time to serve the n_k' units $(k = 1, \ldots, r)$. Then, we have

$$T = \sum_{k=1}^{p-1} T_k' + \sum_{k=1}^{p} T_k + T_0 \tag{3}$$

Taking expected values on the two sides,

$$W_p = E(T) = \sum_{k=1}^{p-1} E(T_k') + \sum_{k=1}^{p} E(T_k) + E(T_0) \tag{4}$$

Now, we have

$$E(T_k) = (1/\mu_k)E(n_k) = (1/\mu_k)(\lambda_k W_k) = \rho_k W_k \tag{5}$$

where we use the notation

$$\rho_k = \lambda_k/\mu_k, \quad k = 1, 2, \ldots, r \tag{6}$$

Also,

$$E(T_k') = (1/\mu_k)\lambda_k E(T) = \rho_k W_p \tag{7}$$

Using the notation

$$\sigma_k = \sum_{i=1}^{k} \rho_i, \quad \sigma_0 = 0; \; k = 1, \ldots, r \tag{8}$$

we have, from (4), on using (5) and (7)

$$W_p = \sigma_{p-1} W_p + \sum_{k=1}^{p} \rho_k W_k + E(T_0) \tag{9}$$

whence, solving for W_p we have

$$W_p = \left[\sum_{k=1}^{p-1} \rho_k W_k + E(T_0) \right] / (1-\sigma_p) \tag{10}$$

Now, we shall prove by induction that the solution of (10) is

$$W_p = \frac{E(T_0)}{(1-\sigma_{p-1})(1-\sigma_p)} \tag{11}$$

For, taking the solution (11) to be true for W_k $(k = 1, \ldots, p-1)$, we have

$$\sum_{k=1}^{p-1} \rho_k W_k = E(T_0) \sum_{k=1}^{p-1} \frac{\rho_k}{(1-\sigma_{k-1})(1-\sigma_k)} \tag{12}$$

$$= E(T_0) \sum_{k=1}^{p-1} \left[\frac{1}{1-\sigma_k} - \frac{1}{1-\sigma_{k-1}} \right], \text{ since } \rho_k = \sigma_k - \sigma_{k-1}$$

$$= E(T_0) \left[\frac{1}{1-\sigma_{p-1}} - 1 \right]$$

Substitution from (12) in (10) shows that (11) holds for W_p. Also, setting $p = 1$ in (10) and noting that an empty sum is zero, we have

$$W_1 = \frac{E(T_0)}{1-\sigma_1} = \frac{E(T_0)}{(1-\sigma_0)(1-\sigma_1)} \tag{13}$$

which shows that (11) holds for W_1. Thus (11) is established.

Now, (13) can be written

$$W_1 = E(T_0)/(1-\rho_1) \tag{14}$$

which gives the mean queueing time of a customer of priority class 1. This should obviously be the same as the mean queueing time W_q in an ordinary M/G/1 queue without priorities, the queue discipline being first-come first-served. This queueing time was shown in 3.3(40) to be

$$W_q = \frac{\lambda E(v^2)}{2(1-\rho)} \tag{15}$$

Comparing (14) and (15) we see that

$$E(T_0) = (\lambda/2)E(v^2) = (\lambda/2) \int_0^\infty v^2 dB(v) \tag{16}$$

the expectation $E(v^2)$ standing for the second moment of the service time distribution about the origin.

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On substitution for $E(T_0)$ from (16) in (11), we have the mean queueing time W_p of the priority class p.

The following direct proof of (16) is interesting.

Measuring time from the commencement of the service in progress, the probability of an arrival in $(t, t+dt)$ is λdt and then the residual service time up to completion is $v-t$, where v is the service time. Also $dB(v)$ is the probability that the service time lies in $(v, v+dv)$. Hence,

$$E(T_0) = \int_{v=0}^{\infty} \int_{t=0}^{v} (v-t)\lambda dt dB(v) \tag{17}$$

$$= (\lambda/2) \int_0^{\infty} v^2 dB(v)$$

which is the same as (16).

It may be noted that for (10) to give a positive value of W_p we should have $\sigma_p < 1$. The above discussion is in fact a steady-state analysis. We conclude that the steady-state will exist if

$$\sigma_r = \rho_1+\rho_2+\ldots+\rho_r < 1 \tag{18}$$

which is obtained by requiring that W_k for $k = 1, 2, \ldots, r$ are all positive.

PROBLEM SET 4

1. For the system $M^X/M/\infty$, show that the steady-state equations are given by

$$0 = -(\lambda+n\mu)P_n+\lambda \sum_{r=1}^{n} P_{n-r}\, a_r+(n+1)\mu P_{n+1}$$

$$0 = -\lambda P_0+\mu P_1$$

Hence show that the p.g.f. $P(z) = \sum_{n=0}^{\infty} P_n z^n$ is given by

$$P(z) = \exp\left[-(\lambda/\mu) \int_z^1 \frac{1-A(u)}{1-u}\, du\right]$$

2. From Problem 1, deduce that, for $M/M/\infty$, $P(z) = \exp[(\lambda/\mu)(z-1)]$, so that $P_n = e^{-\lambda/\mu}(\lambda/\mu)^n/n!$.

3. In Problem 1, by taking the left-hand derivative of $P(z)$ at $z = 1$, show that the mean number in the system is $\lambda\bar{a}/\mu$, where $\bar{a}$ is the mean group size. Also find the variance of the number in the system by differentiating once again.

$[(\lambda/\mu)(\bar{a}+A''(1)/2]$

4. In a bulk arrival system with arrival group size distribution a_r given by 4.1(1), let $p_n(t)$ be the probability that n customers arrive in the interval $(0, t]$; show that

$$\frac{d}{dt} p_n(t) = -\lambda p_n(t)+\lambda \sum_{r=1}^{n} p_{n-r}(t) a_r, \ (n = 0, 1, 2, \ldots)$$

Hence prove that the p.g.f. $P(z, t) = \sum_{n=0}^{\infty} p_n(t) z^n$ is given by

$$P(z, t) = \exp[(A(z)-1)\lambda t].$$

Deduce that the mean number of arrivals is $\lambda\bar{a}t$, $\bar{a}$ being the mean group size. Show also that the variance of the number of arrivals is $\lambda t(\bar{a}+A''(1))$.

[It may be noted that the results of Problem 4 do not depend upon the service time distribution or the number of servers and will, therefore, hold for the system $M^X/G/c$.]

5. In Problem 4, show that from $P(z, t)$ we can obtain the probabilities $p_n(t)$ as

$$p_0(t) = e^{-\lambda t}$$

$$p_n(t) = \left(\frac{\lambda t}{n}\right) \sum_{i=0}^{n-1} (n-i) a_{n-i} p_i(t), \quad (n \geqslant 1)$$

[$p_0(t)$ can be obtained by setting $z = 0$ in $P(z, t)$. To obtain $p_n(t)$, $n \geqslant 1$, differentiate partially w.r.t. z both sides of the equation giving $P(z, t)$ and equate coefficients of z^n on both sides.]

NOTE. The equation $P(z, t) = \exp[(A(z)-1)\lambda t]$ of Problem 4 is an important result in the theory of *Stochastic Processes*. It may be noted that $P(z, t)$ is the probability generating function of the counting distribution of events which occur at time points (epochs) distributed according to a Poisson distribution such that at each occurrence epoch a random number X of events occur with $P(X = r) = a_r$. Such a process is called a *Compound Poisson Process*. In the special case when $a_r = \delta_{r1}$, the Kronecker delta, so that at each epoch *only one* event occurs, we get the *Poisson Process*.

6. For the system $M^R/M/1$, where arrivals are in groups of a fixed size R, write down the steady-state transition equations and hence show that the mean number in the system is $\frac{R+1}{2} \cdot \frac{R\lambda}{\mu-R\lambda}$.

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7. Show that the number of customers in $M^R/M/1$ behaves like the number of phases in $M/E_R/1$. Hence prove that the mean number of phases in the latter (Erlangian service) system is $\frac{R+1}{2} \cdot \frac{\rho}{1-\rho}$, where ρ is the traffic intensity of the system.

[Note that in order to have the mean service time $= 1/\mu$ in the second system, each exponential phase must be completed in an average time $1/R\mu$. Hence the μ of Problem 6 should be changed to $R\mu$, and the result follows by setting $\lambda/\mu = \rho$. (See Eq. 3.8(31).)]

8. In the queueing system $M^X/M/1$, let the arrival group size have the geometric distribution

$$a_r = qp^{r-1},\ q = 1-p,\ (r = 1, 2, \ldots)$$

Show that

$$P_0 = 1-\rho$$

$$P_n = q\rho(1-\rho)(p+q\rho)^{n-1},\ n \geqslant 1$$

Hence obtain the mean and variance of the number in the system.

$$\left[\frac{\rho}{q(1-\rho)},\ \frac{\rho}{q(1-\rho)} + \frac{\rho\{2p+(q-p)\rho\}}{q^2(1-\rho)^2}\right]$$

9. From the results of Problem 8 deduce the corresponding results (for P_n, mean and variance) for the system M/M/1.

[Take $q = 1$. (Why ?)]

10. Consider the following queueing system, which may be termed $M/M^{b-X}/1$ (Cf. Kambo and Chaudhry (1985)) and is applicable to queues for a bus at intermediate stations (different from starting station and destination) :

The server is always busy and on the completion of each service he starts a new service run, taking $b-X$ customers from those waiting, or the whole queue, whichever is less, X being a random variable distributed as

$$P(X = r) = c_r\ (r = 0, 1, 2, \ldots, b)$$

Thus, a service *run* may start taking zero from the queue if none are waiting or if $X = b$. (In the latter case the server is full to capacity.) Define the steady-state probabilities

$$P_n = P\{n \text{ units in queue (at the bus station)}\}$$

Show that the following equations hold :

$$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu \sum_{r=0}^{b} c_{b-r} P_{n+r}\ (n \geqslant 1)$$

$$(\lambda+\mu)P_0 = \mu \sum_{r=0}^{b} C_{b-r} P_r$$

where

$$C_r = \sum_{i=0}^{r} c_i,\quad 0 \leqslant r \leqslant b$$

Using generating function and Rouché's theorem show that when

$$\rho = \lambda/\mu(b-\bar{c}) < 1,\ \left(\bar{c} = \sum_{r=0}^{b} r c_r\right),^*$$

we have

$$P_n = (1-\alpha)\alpha^n\ (n \geqslant 0)$$

where α is the *unique* root, inside the unit circle $|z| = 1$, of the equation

$$\mu z^{b+1} \sum_{r=0}^{b} c_r z^{-r} - (\lambda+\mu)z + \lambda = 0$$

11. Consider and interpret the special case of Problem 10 when $b = 1$.

12. For the queueing system $M^X/D/1$ with constant service time b, show that P_n, the steady-state probabilities for the number in the system, are given by

$$P_n = (P_0+P_1)p_n(b) + \sum_{i=2}^{n+1} P_i p_{n-i+1}(b),\quad (n \geqslant 0).$$

[Proceed as in Section 3.7.]

Note that the above equation can be used recursively to evaluate P_n $(n = 1, 2, \ldots)$ in terms of P_0 and $p_n(b)$, $(n = 0, 1, 2, \ldots)$. Now, the $p_n(b)$ can be evaluated using results of Problem 5 with t replaced by b. And, P_0 can be evaluated from intuitive reasoning by equating the average input and output rates of the system (see, for example, Chaudhry and Templeton (1983), p. 72) as follows :

$$\lambda\bar{a} = (1-P_0)(1/b)$$

whence

$$P_0 = 1-\lambda\bar{a}b = 1-\rho$$

*Note that $(b-\bar{c})$ is the average intake by the server.

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13. Using the equation for P_n given in Problem 12 and the generating function $P(z) = \sum_{n=0}^{\infty} P_n z^n$, and proceeding as in Section 3.3, show that for the $M^X/D/1$ system

$$P(z) = \frac{(1-\rho)(1-z)}{1-z\exp[\lambda b(1-A(z))]}$$

Hence find the mean and variance of the number in the system.

$$\left[\text{Mean} = L = \rho + \frac{\rho^2 + \rho A''(1)/\bar{a}}{2(1-\rho)} \right.$$

$$\text{Variance} = \text{Var}[N] = P''(1-0) + L - L^2$$

where

$$P''(1-0) = \frac{1}{6(1-\rho)^2}\left\{6\rho^2 - 4\rho^3 + \rho^4 + \frac{6\rho A''(1)}{\bar{a}} + \frac{3\rho^2(A''(1))^2}{\bar{a}^2} + 2\rho(1-\rho)\frac{A'''(1)}{\bar{a}}\right\}$$

Cf. 3.3(54), *et seq.* Note that for M/D/1, $A''(1) = A'''(1) = 0$ and hence we have

$$P''(1-0) = \frac{\rho^3(\rho^2 - 4\rho + 6)}{6(1-\rho)^2}$$

which tallies with 3.3(60).]

14. Notice the similarity between the relation given in Problem 13 and Eq. 3.3(28). How would you account for it ?

CHAPTER 5

APPLICATIONS

5.1 Introduction

In this chapter we shall discuss applications of queueing models in problem solving and decision making in various types of human activities. Although appearance of queues or waiting lines (physical or, less discernible, non-physical) is a universal phenomenon, thus making queueing theory applicable in almost every walk of life, ranging from daily routine jobs to complicated production processes, yet there are some areas which have traditionally been taken as particularly suited to such an application. It is these areas like telephone traffic, machine interference, inventories, scheduling, etc., which we shall be discussing *vis-a-vis* application of queueing theory. In fact, several applications are apparent from the examples and problems in the foregoing chapters.

5.2 Two Types of Models

Models, which may be defined as abstractions of real-life phenomena used to facilitate analysis (*e.g.*, a set of differential-difference equations is a mathematical model of a queueing situation), are classified in several ways. One such classification divides models into (a) descriptive, and (b) normative (or prescriptive) models. The former type of models describe the state of affairs of a system on an "as is" basis, while the latter tend to seek a norm or "as it should be". Thus, all "optimization" models are normative. As should be evident from earlier chapters, most of the models used in queueing theory are descriptive in nature inasmuch as most of the time we seek to study the queue length, the waiting time or the busy period through probability distributions. However, consideration of business and industrial applications has brought in the concept of optimization in queues.

Optimization. In optimization studies in queueing theory, *generally* arrivals are taken as *uncontrollable* or *uncontrolled*, and we try to find the optimal service rate or the optimal number of service channels. In such studies we consider two types of costs—(i) the cost of providing service including the cost of the server's idle time, and (ii) the cost of the customer's waiting time. As we increase the service rate (say μ), the first type of cost increases and the second decreases; (see the figure on the next page). We are interested in the lowest point in the total cost curve, which corresponds to

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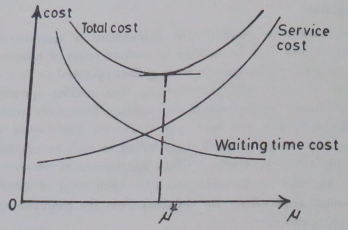
the optimal value of μ, denoted by μ^*. This is the general format of the optimization studies in queueing systems, subject to slight variations depending upon the problem studied. We now proceed to describe applications of the theory of queues in various areas, though such a description can never be exhaustive.



5.3 Applications to Telephone Traffic

As pointed out in Chapter 1, telephone congestion problems accounted for some of the initial applications of queueing theory. We start with some basic concepts in telephony.

A *full availability* system is a set of devices (telephones, trunk lines, etc.), which are all accessible to a group of sources of calls (telephone subscribers). Otherwise, we have a *limited availability* system, where different groups of sources have only a limited number of devices available to them. A hybrid between the two is a *link system*, where linkages exist between groups of sources and destinations with limited access available between other combinations. A *common control* system is a *loss system* where all calls arrive at a central control before going into service. Thus, a call is lost when the control is busy, even if a service channel is free. In a loss system there is no arrangement for a call to wait; it is served on arrival or is lost. Alternatively, we have a *delay system*, in which it is possible for a call to wait until a free channel is available for its service. Most systems studied in queueing theory are thus delay systems.

We shall begin this description of queueing theory applications in telephone systems with :

(a) *Loss Systems.* Let calls arrive in a Poisson fashion with mean rate λ and let the service times (usually called holding times) be exponentially distributed with mean rate μ. Further, let there be c telephone channels

with no provision for waiting, so that a call is lost if it finds all the channels busy on its arrival. Thus, we have the queueing system M/M/c/c, which has already been discussed in Section 1.6(e) with the help of state-dependent parameters. Alternatively, we may proceed by writing the steady-state equations as follows :

$$(\lambda+n\mu)P_n = \lambda P_{n-1}+(n+1)\mu P_{n+1}, \quad 1 \leqslant n \leqslant c-1 \tag{1}$$

$$\lambda P_0 = \mu P_1 \tag{2}$$

$$c\mu P_c = \lambda P_{c-1} \tag{3}$$

where $P_n = P\{n \text{ channels are busy}\}$.

As before, the solution is

$$P_n = \frac{\rho^n/n\,!}{1+\rho+\rho^2/2\,!+\ldots+\rho^c/c\,!}, \quad \rho = \lambda/\mu, \; n = 0, 1, \ldots, c \tag{4}$$

(4) is called *Erlang's loss distribution**. For $n = c$, we have *Erlang's loss formula*. When $c \to \infty$, (4) becomes

$$P_n = e^{-\rho}\rho^n/n\,! \tag{5}$$

giving a Poisson distribution for the number of busy channels, as we observed for the system M/M/∞ (Section 1.6(e)).

If we take the mean arrival rate to be λ_i when i channels are busy ($i = 0, 1, \ldots, c$) and write $\lambda_i/\mu = \rho_i$, then it can be shown as in Section 1.6 that

$$P_n = P_0 \cdot \rho_0\rho_1 \ldots \rho_{n-1}/n\,! \tag{6}$$

where P_0 is obtained from the normalizing condition

$$\sum_{n=0}^{c} P_n = 1 \tag{7}$$

Now, the number of lost calls per unit time is

$$\lambda_c P_c = \mu\rho_c P_c = \mu\rho_0 P_0(\rho_1\rho_2 \ldots \rho_c/c\,!) \tag{8}$$

And, the total number of calls arriving per unit time is

$$\sum_{n=0}^{c} \lambda_n P_n = \mu \sum_{n=0}^{c} \rho_n \cdot P_0 \cdot \frac{\rho_0\rho_1 \ldots \rho_{n-1}}{n\,!} \tag{9}$$

$$= \mu\rho_0 P_0 \sum_{n=0}^{c} \frac{\rho_1\rho_2 \ldots \rho_n}{n\,!}$$

$$= \mu\rho_0 P_0 \left[1+\rho_1+\frac{\rho_1\rho_2}{2\,!}+\ldots+\frac{\rho_1\rho_2 \ldots \rho_c}{c\,!}\right]$$

*Though we have proved (4) for M/M/c/c, the result being independent of the service time distribution holds for the more general system M/G/c/c.

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From (8) and (9), we obtain the proportion of lost calls or the probability that incoming calls are lost as

$$\text{(10)} \qquad \text{P(loss)} = \frac{\rho_1\rho_2 \ldots \rho_c/c\,!}{1+\rho_1+(\rho_1\rho_2/2\,!)+\ldots+(\rho_1\rho_2 \ldots \rho_c/c\,!)}$$

which is independent of ρ_0. If we set $\rho_i = \rho$ $(i = 1, 2, \ldots, c)$, (10) yields the Erlang loss formula.

Assuming calls to originate from a finite set of N sources, and choosing the mean service time as the unit of time so that $\mu = 1$, we can write $\rho_i = (N-i)\lambda$. This yields the *O'Dell distribution* if λ is constant and the *Engset distribution* if λ depends on loss. Substituting for ρ_i in (10), it is easy to verify that the denominator can be expressed in terms of the incomplete beta function ratio :

$$\text{(11)} \qquad (1+\lambda)^{N-1} \cdot \frac{\int_{\lambda/(\lambda+1)}^{1} x^c(1-x)^{N-c-2}\,dx}{\int_0^1 x^c(1-x)^{N-c-2}\,dx}$$

Letting $N \to \infty$, so that $\lambda N = \lambda'$, a constant, (and consequently $\lambda \to 0$), we shall see that in this case (10) will yield the Erlang formula :

$$\text{(12)} \qquad \text{P\{loss\}} = [(\lambda')^c/c\,!] \Big/ \Big[\sum_{i=0}^{c} (\lambda')^i/i\,!\Big]$$

(b) *Loss-Delay Systems.* Let there be c channels servicing calls originating from two input sources with Poisson arrival rates λ_1, λ_2, respectively. The channels have identical exponential service times with parameter μ. We assume that the calls from first source can wait for service, but the second source calls are lost if all the channels are busy.

Let P_n be the steady-state probability that there are n units in the system, *i.e.*, n channels are busy if $n \leqslant c$, and $n-c$ calls are waiting if $n > c$. Then, using state-dependent parameters as in Section 1.6, we have

$$\text{(13)} \qquad \text{'}\lambda_n\text{' of Sec. 1.6} = \begin{cases} \lambda_1+\lambda_2, & n < c \\ \lambda_1, & n \geqslant c \end{cases}$$

$$\text{(14)} \qquad \text{'}\mu_n\text{' of Sec. 1.6} = \begin{cases} n\mu, & n < c \\ c\mu, & n \geqslant c. \end{cases}$$

Proceeding as in Sec. 1.6, we obtain the following steady-state solution [Cohen (1980)]

$$\text{(15)} \qquad P_n = \begin{cases} \dfrac{c-\rho_1}{c-\rho_1+\rho_1E_c(\rho_1+\rho_2)} \cdot \dfrac{(\rho_1+\rho_2)^n/n\,!}{N_c(\rho_1+\rho_2)}, & 0 \leqslant n < c \\ \dfrac{(c-\rho_1)E_c(\rho_1+\rho_2)}{c-\rho_1+\rho_1E_c(\rho_1+\rho_2)} \cdot \dfrac{(\rho_1/c)^{n-c}}{}, & n \geqslant c \end{cases}$$

where

$$\text{(16)} \qquad N_c(\rho) = \sum_{i=0}^{c} (\rho^i/i\,!);\ E_c(\rho) = (\rho^c/c\,!)/N_c(\rho);\ \rho_i = \lambda_i/\mu,\ i = 1, 2$$

The steady-state exists when $\rho_1 < c$.

The probability of a call being lost is

$$\text{(17)} \qquad \text{P}(>0) = \sum_{n=c}^{\infty} P_n = \frac{cE_c(\rho_1+\rho_2)}{c-\rho_1+\rho_1E_c(\rho_1+\rho_2)}$$

The delay (queueing) time density function $f_q(w)$ is given by

$$\text{(18)} \qquad f_q(w)dw = \sum_{n=c}^{\infty} P_n e^{-c\mu w} \cdot \frac{(c\mu w)^{n-c}}{(n-c)\,!} \cdot c\mu dw$$

which simplifies to

$$\text{(19)} \qquad f_q(w) = \text{P}(>0) \cdot \mu(c-\rho_1)e^{-(c-\rho_1)\mu w}$$

The average delay for all calls from the first source is

$$\text{(20)} \qquad W_q = \text{P}(>0)/[(c-\rho_1)\mu]$$

so that the average delay for delayed calls is

$$\text{(21)} \qquad W_q/\text{P}(>0) = 1/[(c-\rho_1)\mu]$$

which is independent of ρ_2.

5.4 Applications to Inventories

Management of inventories is an area where the concepts of queueing theory have an obvious application. A possible model is that in which the units in inventory are regarded as forming a queue, and the arrival of a customer order is likened to a service completion, since such an arrival results in depletion of the queue. The 'units' may arrive in inventory singly or in certain lots of fixed or variable size. In the latter case we have a bulk-arrival queueing system. The time between the arrivals of customer orders

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may be regarded as the service time. The time of 'serving' the customer, *i.e.*, the time spent in issuing the item from inventory may be considered negligible or included in the "service time" defined above.

Though the above assumptions can be considered valid, a better correspondence between an inventory system and a queueing system is obtained by regarding arrivals of customers or customer orders in the inventory system as the 'arrivals' of customers of the queueing system. The reason is that both of these are more or less uncontrollable. The inventory replenishment time or the lead time will then be considered as the service time of the queueing system, and both of these are, in general, controllable by the management of the system. We give below some interesting applications.

(a) *The Multi-Channel Queue Model.* Let us keep a certain number c of units in stock and order for replenishment each time an item is sold from stock. The time taken in effecting a sale from available stock is negligible. Let the customer orders for sale arrive at a mean rate λ and let the average replenishment time be $1/\mu$. Further, assume the inter-arrival time of customer orders to be exponential and the replenishment time distribution to be k-Erlang, and that any customer orders arriving when there are no items in stock are lost. The system may thus be regarded as a queueing system with c channels. The arrival of a customer order results in the depletion of the stock by one unit and the placing of a replenishment order. Thus we may regard the customer order as the arrival of a unit to the queueing system, an empty space in inventory as a busy channel, and the replenishment time as the service time. The inventory system under consideration may, therefore, be regarded as an $M/E_k/c/c$ system with no queues formed. Note that in this system the number in the system is equal to the number of busy channels, which is the number of replenishment orders outstanding in the inventory system. So, the (steady-state) probability that n replenishment orders are outstanding is given by

$$P_n = (\rho^n/n\,!) \Big/ \sum_{j=0}^{c} (\rho^j/j\,!), \quad n = 0, 1, 2, \ldots, c \tag{1}$$

where $\rho = \lambda/\mu$. The mean number of outstanding replenishment orders is

$$L = \sum_{n=1}^{c} nP_n = \rho S(\rho, c-1)/S(\rho, c) \tag{2}$$

where

$$S(\rho, c) = \sum_{j=0}^{c} [e^{-\rho}\rho^j/j\,!] \tag{3}$$

The functions $S(\rho, c)$ can be evaluated with the help of Poisson distribution tables.

The mean number of sales per unit time is

$$\begin{aligned} L_s = \lambda(1-P_c) &= \lambda \sum_{j=0}^{c-1} (\rho^j/j\,!) \Big/ \sum_{j=0}^{c} (\rho^j/j\,!) \\ &= \lambda S(\rho, c-1)/S(\rho, c) = \mu L \end{aligned} \tag{4}$$

Since an order is placed every time a sale is made from stock, the mean number of orders per unit time is equal to the mean number of sales per unit time, *i.e.*, L_s or μL. Also average inventory is equal to the average number of idle channels $= c-L$.

Let G be the gain or profit made per sale, *i.e.* the selling price minus the purchase price, C_p the cost of procurement or the ordering cost, and C_i the inventory carrying cost per unit time. Then, the average net profit per unit time is

$$\begin{aligned} P(c) &= G\cdot\mu L - C_p\cdot\mu L - C_i(c-L) \\ &= (g\mu + C_i)L - C_i c \\ &= (g\mu + C_i)\cdot\rho S(\rho, c-1)/S(\rho, c) - C_i c \end{aligned} \tag{5}$$

where $g = G-C_p$. If c^* is the optimal value of the maximum stock c, we should have

$$P(c^*) > P(c^*-1) \text{ and } P(c^*) > P(c^*+1) \tag{6}$$

Therefore, c^* is given by

$$T(\rho, c^*) - T(\rho, c^*-1) > C_i/\{(g\mu + C_i)\rho\} > T(\rho, c^*+1) - T(\rho, c^*) \tag{7}$$

where

$$T(\rho, c) = S(\rho, c-1)/S(\rho, c) \tag{8}$$

An easy approach to find c^* is to look for the smallest c^* which satisfies the inequality

$$P(c^*+1) < P(c^*) \tag{9}$$

which gives

$$T(\rho, c^*+1) - T(\rho, c^*) < C_i/(g\mu + C_i)\rho \tag{10}$$

It is worth noting that instead of considering the profit as we did in writing Eq. (5), we could have written the total cost equation as is usual in inventory studies. Thus, the average inventory carrying cost per unit time is C_i times the average number of items in stock, *i.e.* $C_i(c-L)$ and the ordering cost per unit time is $C_p\cdot\mu L$. Also, average number of lost sales per unit time is

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$\lambda P_c = \lambda - \mu L$, from (4). Therefore, the shortage cost per unit time, given by the loss of profit, is $G(\lambda - \mu L)$. Hence, the total relevant cost per unit time is

$$T.C. = C_t(c-L) + C_p \mu L + G(\lambda - \mu L) = G\lambda - P(c) \tag{11}$$

where $P(c)$ is given by (5).

Since $G\lambda$ is constant (independent of c), it is clear that maximization of the profit $P(c)$ and minimization of the cost $T.C.$ are equivalent.

(b) *Re-order on Depletion of Stock.* Next, we consider the model where we start with a stock of c items and order for replenishment when the inventory is reduced to zero, *i.e.* when all the units in stock are sold out. Let customer orders arrive in a Poisson fashion at mean rate λ and let the replenishment time have a general distribution with mean $1/\mu$. The c units in inventory will last an average duration c/λ and the average duration of the out-of-stock situation will be $1/\mu$, the average replenishment time. So, in the steady state the system will be out of stock for a fraction of time $(1/\mu)/[(c/\lambda)+(1/\mu)]$. It will be convenient for this model to define P_n to be the probability that there are n units in stock. So the probability of being out of stock will be

$$P_0 = \rho/(c+\rho), \quad \text{where } \rho = \lambda/\mu \tag{12}$$

For the remaining fraction of time $c/(c+\rho)$, the system behaves like a pure death process with the transition equations :

$$\begin{cases} \lambda P_n = \lambda P_{n+1}, & n = 1, 2, \ldots, c-1 \\ \mu P_0 = \lambda P_1 \\ \lambda P_c = \mu P_0 \end{cases} \tag{13}$$

From Eqs. (13), we have

$$P_1 = P_2 = \ldots = P_c = 1/(c+\rho) \tag{14}$$

Therefore, the mean number of units in stock is

$$\sum_{n=1}^{c} nP_n = c(c+1)/2(c+\rho) \tag{15}$$

Mean number of sales per unit time is

$$\lambda(1-P_0) = \lambda c/(c+\rho) \tag{16}$$

Also, with probability P_0 only one replenishment order is issued during one average replenishment time $1/\mu$. Therefore, the mean number of orders issued per unit time is

$$\mu P_0 = \lambda/(c+\rho) \tag{17}$$

Therefore, the average profit per unit time as a function of c is

$$P(c) = [G\lambda c - C_p \cdot \lambda - C_t c(c+1)/2]/(c+\rho) \tag{18}$$

From (18) we can find the optimal value of c by using methods for maximizing a function of a discrete variable. [Cf. Eqs. (5) through (7)].

For an approximate solution, however, we can proceed as follows. Regard c as a continuous variable and replace $c(c+1)$ by c^2. This will be justified if c is considered large. Then, setting the derivative $P'(c)$ equal to zero we see that the optimal value c^* should satisfy

$$(c+\rho)(G\lambda - C_t c) - [G\lambda c - C_p \cdot \lambda - C_t c^2/2] = 0$$

or,

$$c^2 + 2\rho c - 2\lambda(G\rho + C_p)/C_t = 0 \tag{19}$$

which yields

$$c^* = \sqrt{\rho^2 + 2\lambda(G\rho + C_p)/C_t} - \rho \tag{20}$$

It is interesting to note that if we take the replenishment time to be zero, so that $\rho = 0$, Eq. (20) yields

$$c^* = \sqrt{2\lambda C_p/C_t} \tag{21}$$

which is the same as the Wilson formula for the economic order quantity (E.O.Q.). This simplification of the above result could be expected, since in the above approximate solution we are regarding c as a continuous variable, the demand rate is uniform and no shortage is allowed. Also, in this case the term containing G in (18), which represents profit made on sales, will not contribute anything to the solution, since in this case no sales are lost. Thus the model discussed reduces to the usual E.O.Q. model.

It may be noted that the policy of ordering for a single unit for each item sold will be suitable in case of bulky and expensive items, so that inventory holding cost is high compared to ordering cost, whereas the policy of ordering for all the c items stored on (or close to) depletion of stock may be considered advisable where ordering and handling costs are high in comparison to inventory carrying and shortage costs.

(c) *Use of a Re-Order Level.* We now consider an improvement of the model discussed in (*b*) above, where a replenishment order for Q items is sent whenever the inventory level comes down to D (the re-order level), so that the maximum inventory level for the system is $Q+D = M$, say. As before, customer orders are assumed to arrive in a Poisson fashion with mean rate

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λ and the replenishment times are taken to be exponentially distributed with mean $1/\mu$. The steady-state equations for this system are

$$(22)\qquad \begin{aligned} \lambda P_M &= \mu P_{M-Q} = \mu P_D && \\ \lambda P_n &= \mu P_{n-Q} + \lambda P_{n+1} && (Q \leqslant n < M) \\ \lambda P_n &= \lambda P_{n+1} && (D < n < Q) \\ (\lambda+\mu) P_n &= \lambda P_{n+1} && (0 < n \leqslant D) \\ \mu P_0 &= \lambda P_1 && \end{aligned}$$

The above equations yield the following solutions :

$$(23)\qquad \begin{aligned} P_n &= (\mu/\lambda)\{(\lambda+\mu)/\lambda\}^{n-1} P_0 && (0 < n \leqslant D) \\ &= (\mu/\lambda)\{(\lambda+\mu)/\lambda\}^{D} P_0 && (D < n \leqslant Q) \\ &= (\mu/\lambda)[\{(\lambda+\mu)/\lambda\}^{D} - \{(\lambda+\mu)/\lambda\}^{n-Q-1}] P_0, && (Q < n \leqslant M) \end{aligned}$$

where

$$(24)\qquad P_0 = \left[1 + \frac{Q\mu}{\lambda}\left(\frac{\lambda+\mu}{\lambda}\right)^{D}\right]^{-1} = \lambda^{D+1}/\{\lambda^{D+1} + Q\mu(\lambda+\mu)^{D}\}$$

Therefore, the mean number of units in stock is

$$(25)\qquad I = \sum_{n=1}^{M} n P_n = [(\lambda+\mu)^{D}\{(Q+2D+1)\mu - 2\lambda\}/2 + \lambda^{D+1}]Q/[\lambda^{D+1} + Q\mu(\lambda+\mu)^{D}]$$

Mean number of sales per unit time is

$$(26)\qquad S = \lambda(1 - P_0) = Q\lambda\mu(\lambda+\mu)^{D}/[\lambda^{D+1} + Q\mu(\lambda+\mu)^{D}]$$

And, as in Eq. (17), the mean number of replenishment orders issued per unit time is

$$(27)\qquad R = \mu \sum_{n=0}^{D} P_n = \lambda\mu(\lambda+\mu)^{D}/[\lambda^{D+1} + Q\mu(\lambda+\mu)^{D}]$$

Hence, with the notation of Eq. (18), the expected profit per unit time is

$$(28)\qquad P(Q, D) = GS - C_I I - C_p R$$

which is a function of two discrete variables Q and D, with S, I, and R given by (25)–(27). For an approximate solution for optimal values of Q and D, we can differentiate w.r.t. Q and D and equate the derivatives to zero. This gives two equations to determine optimal Q and D.

However, an alternative method of finding the value of D, which includes a "buffer" stock, is to fix P_0, the probability of being out of stock at a pre-assigned value p, which can be chosen by management and then obtain, from (24),

$$(29)\qquad D = \log\{(\lambda/Q\mu)(1/p - 1)\}/\log(1 + \mu/\lambda)$$

It may be noted that (29) expresses D as the ratio of two logarithms, so that the base of the logarithms is immaterial; natural logarithms or common logarithms may be used as convenient.

For another approximate analysis, we may proceed as follows. For small values of P_0, we should have $\{\lambda/(\lambda+\mu)\}^{D} = h$, say, a small quantity. Then, in (25)–(27), I, S, and R can be expressed in powers of h, say up to the first power of h. These values can then be used in (28), thus simplifying the analysis considerably. As a first approximation, if we set $h = 0$, it can be verified that the equation $\dfrac{\partial}{\partial Q} P(Q, D) = 0$ yields

$$(30)\qquad Q = \sqrt{2\lambda C_p / C_I}$$

showing that even for this complicated model the Wilson formula gives an approximate value for the optimal order quantity Q^*. However, a better approximation to Q^* can be easily obtained containing a *correction term* as a multiple of h.

5.5 Applications to Machine Interference

Machine interference problems are typical of textile industry, where a number of automatic machines, like thread-cutting or wire-drawing machines, are put under the charge of a single operator. From time to time a machine may stop and require a corrective action by the operator, after which it again starts working. Another example of this type is that of workers, who from time to time have to stop their work in order to grind their tools at a grinding machine. If at any time more than one machine need the operator's attention or more than one workman report for grinding tools, a queue develops. One may like to find the optimal number of machines to be assigned to an operator or the optimal number of workmen assigned a grinding machine.

The queueing model corresponding to the above situations, where the arrivals are generated by a finite source, has already been discussed in Section 1.6(f). Let k machines be under the supervision of a single operator, whose service time is exponential with mean $1/\mu$, and let the stoppages of any machine occur in a Poisson fashion with mean rate λ per unit time; then it was shown in 1.6(42)–(44) that the steady-state probability of finding n machines failed is given by

$$(1)\qquad P_n = \{k\,!/(k-n)\,!\}(\lambda/\mu)^{n} P_0$$

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where

$$P_0 = 1/\left[\sum_{j=0}^{k} \{k\,!/(k-j)\,!\}(\lambda/\mu)^j\right]. \tag{2}$$

From (1) and (2) we can write

$$P_n = \frac{e^{-\mu/\lambda}(\mu/\lambda)^{k-n}/(k-n)\,!}{\sum_{j=0}^{k} e^{-\mu/\lambda}(\mu/\lambda)^{k-j}/(k-j)\,!}, \quad (n = 0, 1, 2, \ldots, k) \tag{3}$$

which shows that $k-n$, the number of machines running, has a truncated Poisson distribution with parameter μ/λ. Thus the mean number of machines running is

$$\sum_{n=0}^{k-1}(k-n)P_n = (\mu/\lambda)\cdot\frac{\sum_{j=0}^{k-1} e^{-\mu/\lambda}(\mu/\lambda)^{k-1-j}/(k-1-j)\,!}{\sum_{j=0}^{k} e^{-\mu/\lambda}(\mu/\lambda)^{k-j}/(k-j)\,!} \tag{4}$$

$$= (\mu/\lambda)\,\frac{P(\mu/\lambda;\, k-1)}{P(\mu/\lambda;\, k)}$$

the last expression being in terms of the cumulative Poisson distribution given by

$$P(m;\, r) = \sum_{j=0}^{r} [e^{-m}m^j/j\,!]. \tag{5}$$

Dividing the expression obtained in (4) by k, we get the expected proportion of machines running, which can be maximized as a function of k to give the optimal value of k, when λ and μ are known.

Next, consider the case of k machines under the supervision of c operators. Then we have the model described in Problem 9 of Problem Set 1 (with N replaced by k), *i.e.*

$$\lambda_n = (k-n)\lambda, \quad 0 \leqslant n < k \tag{6}$$
$$= 0, \quad n \geqslant k$$

$$\mu_n = n\mu, \quad 1 \leqslant n < c \tag{7}$$
$$= c\mu, \quad n \geqslant c$$

We assume that $c < k$. Then, proceeding as in Section 1.6, we have

For $0 \leqslant n < c$

$$P_n = \frac{k\lambda.(k-1)\lambda \ldots (k-n+1)\lambda}{\mu.2\mu \ldots n\mu} P_0 = \binom{k}{n}\left(\frac{\lambda}{\mu}\right)^n P_0 \tag{8}$$

and, for $c \leqslant n \leqslant k$

$$P_n = \frac{k\lambda.(k-1)\lambda \ldots (k-n+1)\lambda}{\mu.2\mu \ldots (c-1)\mu.c\mu.(c\mu)^{n-c}} P_0 = \binom{k}{n}\frac{n\,!}{c\,!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0 \tag{9}$$

where

$$P_0 = \left[\sum_{n=0}^{c-1}\binom{k}{n}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{k}\binom{k}{n}(n\,!/c\,!\,c^{n-c})\left(\frac{\lambda}{\mu}\right)^n\right]^{-1} \tag{10}$$
$$= 1/F(k), \quad \text{say}$$

As before, the mean number of machines running is given by

$$\sum_{n=0}^{k-1}(k-n)P_n = kF(k-1)/F(k) \tag{11}$$

$$= \frac{(k\mu/\lambda)\left[\sum_{j=0}^{c-1}\binom{k-1}{j}\left(\frac{\mu}{\lambda}\right)^{k-j-1} + \sum_{j=c}^{k-1}\binom{k-1}{j}\frac{j\,!}{c\,!\,c^{j-c}}\left(\frac{\mu}{\lambda}\right)^{k-j-1}\right]}{\sum_{j=0}^{c-1}\binom{k}{j}\left(\frac{\mu}{\lambda}\right)^{k-j} + \sum_{j=c}^{k}\binom{k}{j}\frac{j\,!}{c\,!\,c^{j-c}}\left(\frac{\mu}{\lambda}\right)^{k-j}}$$

$$= (k\mu/\lambda)\cdot\frac{\left(\frac{\lambda+\mu}{\lambda}\right)^{k-1}B\left(k-1,\, \frac{\lambda}{\lambda+\mu};\, c-1\right) + \frac{(k-1)\,!\,e^{c\mu/\lambda}}{c\,!\,c^{k-c-1}}P(c\mu/\lambda;\, k-c-1)}{\left(\frac{\lambda+\mu}{\lambda}\right)^{k}B\left(k,\, \frac{\lambda}{\lambda+\mu};\, c-1\right) + \frac{k\,!\,e^{c\mu/\lambda}}{c\,!\,c^{k-c}}P(c\mu/\lambda;\, k-c)}$$

where $P(.\,;.)$ is given by (5) and B denotes the cumulative binomial distribution given by

$$B(n, p;\, r) = \sum_{j=0}^{r}\binom{n}{j}p^j(1-p)^{n-j} \tag{12}$$

It can be seen that for $c = 1$, the final result in (11) simplifies to (4). Also, for the case $c = k$, (11) yields the mean number of machines running $= k\mu/(\lambda+\mu)$, and hence the mean number of failed machines $= k\lambda/(\lambda+\mu)$. These results follow also from the fact that for $c = k$, the model discussed above becomes the brand-share model of Section 1.6(g).

Further, from Eq. (8)–(10), we find that the mean number of busy channels or the mean number of machines being repaired is

$$\sum_{n=0}^{c-1} nP_n + \sum_{n=c}^{k} cP_n \tag{13}$$

$$= \sum_{n=1}^{c} n\binom{k}{n}\left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=c+1}^{k} c\binom{k}{n}\frac{n\,!}{c\,!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \left(\frac{k\lambda}{\mu}\right)P_0\left[\sum_{n=0}^{c-1}\binom{k-1}{n}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{k-1}\binom{k-1}{n}\frac{n\,!}{c\,!\,c^{n-c}}\left(\frac{\lambda}{\mu}\right)^n\right]$$

$$= \left(\frac{k\lambda}{\mu}\right)F(k-1)/F(k)$$

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Using (11) and (13), we have the mean number of machines *in the service system*, being repaired or waiting, given by

$$L = k - kF(k-1)/F(k) = k[1-\{F(k-1)/F(k)\}] \tag{14}$$

and the mean number waiting for repair, given by

$$\begin{aligned} L_q &= L - (k\lambda/\mu)F(k-1)/F(k) \\ &= k[1-\{1+(\lambda/\mu)\}\{F(k-1)/F(k)\}] \end{aligned} \tag{15}$$

Also, the mean arrival rate to the system is

$$\lambda' = \sum_{n=0}^{k} (k-n)\lambda P_n = (k-L)\lambda, \quad \text{since } \sum_{n=0}^{k} nP_n = L \tag{16}$$

Therefore, the mean time spent in system is

$$W = L/\lambda' = L/[(k-L)\lambda] \tag{17}$$

and mean time in queue is

$$W_q = L_q/\lambda' = L_q/[(k-L)\lambda] \tag{18}$$

It can be easily verified that

$$W - W_q = 1/\mu \tag{19}$$

as expected.

5.6 Applications to Scheduling

The problem of scheduling jobs for processing at a machine or machine centre out of a waiting line of jobs is just an example of a queueing problem and, therefore, most of the results of previous chapters are applicable to scheduling. Here the *customers* or *units* requiring service are the *jobs* and the *service* operation is the *processing* of the jobs at the machine (centre). In the theory of scheduling, the main emphasis is generally on *scheduling rules* (corresponding to *queue disciplines* in queueing theory) and their effect on the *flow time* of jobs through the machine or machines. Flow time means the waiting (queueing) time plus the processing time which, in a queueing system is called the *total time in the system*.

Scheduling Rules

Some of the scheduling rules used in practice depend upon the processing time, like the *shortest processing time* (SPT) rule according to which the jobs are processed in ascending order of the processing times required by them. As against this, there are three important rules which are independent of processing times : these are the *first-come first-served* (FCFS), the *last-come first-served* (LCFS) and the *random selection* or *service in random order*

(SIRO) rules. Of these, the FCFS discipline has been discussed at length in earlier chapters. We have seen that in an M/M/1/FCFS system, with usual notation the mean time in the queue, W_q, is given by [Cf. 1.5(28)]

$$W_q = \lambda/\{\mu(\mu-\lambda)\} = \rho/\{\mu(1-\rho)\} \tag{1}$$

Proceeding as in 1.5(28), we have the second moment of the queueing time about the origin given by

$$E(w_q^2) = \int_{0+}^{\infty} w^2 . \rho\mu(1-\rho)e^{-(1-\rho)\mu w}\, dw = \frac{2\rho}{\mu^2(1-\rho)^2} \tag{2}$$

Let us now consider the M/M/1/LCFS system. Suppose a job arrives when the server is busy. Then, this job's queueing time will consist of the duration of the residual processing time of the job being processed *plus* the processing times of all jobs that arrive during this processing time and of all subsequent arrivals until the server first becomes free of all subsequent arrivals. Thus, referring to the paragraph just after 2.3(11), we see that the queueing time of a job arriving *during a busy period* is the same as an *ordinary busy period* whose density function has the Laplace transform α_1. Now, since the queue length distribution does not depend on the queue discipline, it follows that under the LCFS discipline, too, a job arrives during an idle period with probability $1-\rho$, in which case his queueing time will be zero. Hence the queueing time density function is given by

$$f_{qL}(w) = (1-\rho)\delta(w) + \rho . \mathscr{L}^{-1}(\alpha_1) \tag{3}$$

so that the Laplace transform of the density function is

$$\bar{f}_{qL}(s) = 1-\rho+\rho\alpha_1 \tag{4}$$

where $\delta(\cdot)$ is the Dirac delta function and α_1 is given by (see 2.3(10))

$$\alpha_1 = [s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}]/(2\lambda) \tag{5}$$

and the subscript L signifies the LCFS discipline.

From the above it follows that the mean queueing time under LCFS discipline is

$$\begin{aligned} W_{qL} &= -\bar{f}'_{qL}(0) = -\frac{\rho}{2\lambda}\left[1-\tfrac{1}{2}.\frac{2(\lambda+\mu)}{\sqrt{(\lambda+\mu)^2-4\lambda\mu}}\right] \\ &= -\frac{1}{2\mu}\left[1-\frac{\lambda+\mu}{\mu-\lambda}\right], \text{ assuming } \mu > \lambda \\ &= \lambda/[\mu(\mu-\lambda)] \end{aligned} \tag{6}$$

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Also, the second moment about the origin of the queueing time is

$$E(w_{qL}^2) = \tilde{f}_q{}''(0) = \frac{2\lambda}{(\mu-\lambda)^3} = \frac{2\rho}{\mu^2(1-\rho)^3} \tag{7}$$

Comparing (6), (7) with (1), (2), we see that

$$W_{qL} = W_q \tag{8}$$

$$E(w_{qL}^2) = \frac{1}{1-\rho} \cdot E(w_q^2) \tag{9}$$

that is, the mean queueing time under LCFS is the same as that under FCFS, but the second moment in the former case is greater than that in the latter by a factor $1/(1-\rho)$. It follows that the queueing time in LCFS has the same mean as in FCFS but a larger *variance*.

For random selection for service, the reader may refer to Saaty (1961), p. 243 *et seq.*, or to Riordan (1962), pp. 103–106. It can be shown that for this system, M/M/1/SIRO, the first two moments are as follows :

$$W_{qR} = W_q \tag{10}$$

$$E(w_{qR}^2) = \frac{1}{1-\rho/2} \cdot E(w_q^2) \tag{11}$$

where R stands for random service. Relations (10) and (11) hold for the multichannel system, too.

Non-preemptive priorities were discussed in Section 4.5. For an analysis of various forms of preemptive priority scheduling rules, one may refer to Jaiswal (1968).

5.7 Applications to Transportation

Transportation systems, being basically service systems, are full of application areas for queueing theory. In what follows we consider two examples of such applications.

(a) *The Double-Ended Queue*

Consider the double-ended queue of customers (passengers) and servers (taxis) at a taxi-stand [Kashyap (1965a, b; 1966), Srivastava and Kashyap (1982)]. We assume that passengers arrive at the taxi stand in a Poisson fashion with mean rate λ and form a queue if no taxis are available. Taxis also arrive at the stand in a Poisson stream with mean rate μ, and pick up one customer each from the queue, if any, or else form a queue. Thus at any time there is either a queue of passengers or one of taxis or of neither. Let

P_n denote the steady-state probability that n passengers are waiting in the queue. We allow n to assume negative values, so that

(i) when $n > 0$, it denotes the numbers of passengers waiting

(ii) when $n = 0$, neither passengers nor taxis are waiting

(iii) when $n < 0$, it signifies that $-n$ taxis are waiting.

Assuming a limited waiting space for N passengers or M taxis, we allow n to vary from $-M$ to N. The steady-state transition equations are

$$0 = -\lambda P_{-M} + \mu P_{-M+1} \tag{1}$$

$$0 = -(\lambda+\mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \quad (-M < n < N) \tag{2}$$

$$0 = -\mu P_N + \lambda P_{N-1} \tag{3}$$

Solving Eq. (1) through (3) recursively as in Section 1.5, we have

$$P_n = \rho^{M+n} P_{-M} \quad (n = -M, -M+1, \ldots, N) \tag{4}$$

where $\rho = \lambda/\mu$. Now, using the normalizing condition

$$\sum_{-M}^{N} P_n = 1 \tag{5}$$

we have

$$P_{-M} = \frac{1-\rho}{1-\rho^{M+N+1}}, \quad \rho \neq 1 \tag{6}$$

so that (4) yields

$$P_n = \frac{(1-\rho)\rho^{M+n}}{1-\rho^{M+N+1}}, \quad \rho \neq 1, (-M \leqslant n \leqslant N) \tag{7}$$

Proceeding to limits as $\rho \to 1$ we have

$$P_n = \frac{1}{M+N+1}, \text{ when } \rho = 1, (-M \leqslant n \leqslant N) \tag{8}$$

The probability that the taxi-stand is empty is

$$P_0 = \frac{(1-\rho)\rho^M}{1-\rho^{M+N+1}}, \quad \rho \neq 1 \tag{9}$$

$$= \frac{1}{M+N+1}, \quad \rho = 1$$

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The probability that there is a queue of passengers (customers) is

$$P_C = \sum_{1}^{N} P_n = \frac{(1-\rho^N)\rho^{M+1}}{1-\rho^{M+N+1}}, \quad \rho \neq 1 \tag{10}$$

$$= \frac{N}{M+N+1}, \qquad \rho = 1$$

The probability that there is a queue of taxis is

$$P_T = \sum_{-M}^{-1} P_n = \frac{1-\rho^M}{1-\rho^{M+N+1}}, \quad \rho \neq 1 \tag{11}$$

$$= \frac{M}{M+N+1}, \qquad \rho = 1$$

The mean queue length of passengers (customers) is

$$L_C = \sum_{1}^{N} nP_n \tag{12}$$

$$= \frac{\rho^{M+1}\{1-(N+1)\rho^N+N\rho^{N+1}\}}{(1-\rho)(1-\rho^{M+N+1})}, \quad \rho \neq 1$$

$$= \frac{N(N+1)}{2(M+N+1)}, \qquad \rho = 1$$

The mean queue length of taxis is

$$L_T = \sum_{-M}^{-1} (-n)P_n \tag{13}$$

$$= \frac{M-(M+1)\rho+\rho^{M+1}}{(1-\rho)(1-\rho^{M+N+1})}, \quad \rho \neq 1$$

$$= \frac{M(M+1)}{2(M+N+1)}, \quad \rho = 1$$

One may be interested in finding the probabilities that passengers or taxis are 'lost' to the system due to the limited waiting space. Clearly, we have

$$P\{\text{passengers are lost}\} = P_N \tag{14}$$

$$P\{\text{taxis are lost}\} = P_{-M} \tag{15}$$

Also,

(16) the mean number of passengers lost per unit time $= \lambda P_N$

and

(17) the mean number of taxis lost per unit time $= \mu P_{-M}$

where P_N and P_{-M} are given by (7), (8).

Waiting Times

Clearly, a customer does not have to wait if he finds a queue of taxis (which occurs with probability P_T). Otherwise, if he finds n customers present, he has to wait until $(n+1)$ taxis have arrived. Hence, the density function of the customers' waiting time is given by

$$w_C(x) = P_T\delta(x) + \sum_{n=0}^{N} P_n \cdot \frac{\mu^{n+1}x^n e^{-\mu x}}{n!} \tag{18}$$

where $\delta(x)$ is the Dirac delta function, and P_n, P_T are given by (7) and (11), respectively. Thus, the mean waiting time for customers is

$$W_C = \int_0^\infty x w_C(x)dx \tag{19}$$

$$= \sum_{n=0}^{N} \int_0^\infty \frac{(1-\rho)\rho^{M+n}}{1-\rho^{M+N+1}} \cdot \frac{(\mu x)^{n+1}e^{-\mu x}}{n!}\, dx$$

$$= \sum_{n=0}^{N} \frac{(1-\rho)\rho^{M+n}}{1-\rho^{M+N+1}} \cdot \frac{n+1}{\mu}$$

where we have interchanged the order of summation and integration, which can be shown to be justified. Eq. (19) simplifies to

$$W_C = \frac{1}{\mu\rho} \cdot \frac{\rho^{M+1}[1-(N+1)\rho^N+N\rho^{N+1}]}{(1-\rho)(1-\rho^{M+N+1})} \tag{20}$$

$$= \frac{1}{\lambda} L_C$$

or,

$$L_C = \lambda W_C \tag{21}$$

which is Little's equation. Proceeding similarly, we can obtain the density function of the taxis' waiting time and show that

$$L_T = \mu W_T \tag{22}$$

where W_T is the mean waiting time for taxis. This is Little's equation for taxis.

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(b) *Queues with Variable Capacity*

Let us consider queues of passengers waiting for buses at an intermediate station like Kingston, Ontario when buses ply between Montreal, Québec and Toronto, Ontario. The intake capacity of the bus will be $b-X$, where b is its maximum capacity (total number of passenger seats) and X is a random variable denoting the number of seats already occupied on arrival at Kingston (after the passengers with Kingston destination have disembarked). Let the customer and bus arrivals have Poisson distributions with mean rates λ and μ, respectively; so the system operates like an $M/M^{b-X}/1$ model (Cf. Kambo and Chaudhry (1985)).

The mathematical formulation and solution of the queue length process for the above model are given in Chapter 4, Problem 10, where it is shown that the steady state number of passengers queueing up for service has a geometric distribution.

5.8 Other Applications

It is a common experience to find queues or waiting lines developing in all spheres of human activity, so that an enumeration of the areas of application of queueing theory can never be exhaustive. For specific application areas the reader may refer to more detailed application-oriented books. For example, one may refer to Lee (1966) for a very good account of applications in air traffic, to Panico (1969) for applications in hospital administration, to Morse (1958) for applications to inventories and maintenance, to Prabhu (1965a) for applications to inventories and storage problems, to Hillier and Lieberman (1986) for applications to industrial queues, to Kleinrock (1976) and Allen (1978) for applications to computer science, to Newell (1982) for applications to transportation queues, and to various books on the subject like Saaty (1961), Cox and Smith (1961), and others for miscellaneous applications.

Already in its eighties, queueing theory continues to remain a growing field. "May its tribe increase !"

PROBLEM SET 5

1. In a large community, people have been observed to go on foreign travel in a Poisson fashion and stay abroad for a duration having an approximately exponential distribution. On an average, 1000 persons leave on any single day and the average duration of their stay abroad is 5 days. Regarding persons on foreign travel at any time as forming an $M/M/\infty$ system, find the steady-state number of persons abroad at any time.

[5000]

2. The telephone calls in a community appear to be generated in a Poisson fashion at the average rate of 5 calls per minute. If the distribution of the duration of a call is observed to be approximately exponential with a mean of 2 minutes, find the minimum number of channels that should be provided in order to ensure a steady state. If 15 channels are provided, find the probability that a call is lost due to all channels being busy. Also find the average number of busy channels.

[11 channels; 0·037; 9·63]

3. Machines arrive at a repair shop in a Poisson fashion at an average rate λ. The machine downtime cost is C_1 per machine per unit time and the cost of providing repair service at a rate μ is $C_2\mu$ per unit time. Taking the service times to be exponential, show that the optimal value of μ is $\lambda+\sqrt{\lambda C_1/C_2}$.

[*Hint.* The total cost of waiting and providing service per unit time is

$$C_1 L + C_2\mu = C_1 \cdot \frac{\lambda}{\mu-\lambda} + C_2\mu \ ]$$

4. A store dealing in costly items follows the inventory policy of carrying a few items in stock and ordering for replenishment as soon as an item is sold from stock. The cost of the item is \$ 10,000 and the carrying cost is reckoned to be 20% per year. The management believe that the shortage cost is \$ 2,000 per unit short per month. The demand is seen to be approximately Poisson with an average rate of 2 units per month and the replenishment lead time can be considered to be exponential with a mean of one month. Considering the system as an M/M/c queueing system, find the optimal stock to be carried.

[5 items]

[If c items are carried in stock, the total expected cost for the lead time duration of 1 month is

$$TC(c) = \$10{,}000\times c\times(0{\cdot}20/12)+\$2{,}000\times L_q$$

$$= \$500c/3+\$2{,}000\times\frac{2(2)^c}{(c-1)!(c-2)^2}\times\left[\sum_{i=0}^{c-1}\frac{2^i}{i!}+\frac{2^c}{c!}\times\frac{1}{1-2/c}\right]^{-1}$$

(using Eq. 1·6(24) for L_q)

Note that for existence of steady state we should have $\lambda/c\mu = 2/c < 1$, so that $c > 2$.

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Calculating $TC(c)$ for a few values, we see that

$TC(3) = \$2{,}277{\cdot}77$

$TC(4) = \$1{,}013{\cdot}83$

$TC(5) = \$\ \ 912{\cdot}60$

$TC(6) = \$1{,}018{\cdot}02$

Thus optimal stock to carry = 5 items.]

5. Assume the data as given in Problem 4 except that the management have no precise estimate of the shortage cost. Given that the company policy is to carry a stock of 6 items, compute the shortage cost, C, imputed* by the company policy.

[$\$5{,}412 < C < \$23{,}487$]

[Write $TC(c)$ with \$2,000 replaced by C. Now, we require that $TC(6) < TC(5)$, and $TC(6) < TC(7)$.]

6. In a community of 100 individuals, persons switching into a brand and those switching out of it are observed to have approximately Poisson distributions with mean rates of 30 and 20 per month. Considering the steady state of the switching behaviour, find

(a) the mean and variance of the brand share

(b) the probability of finding exactly half the community using the brand

(c) the probability of extinction of the brand

(d) the probability of the whole community being saturated with the brand.

[(a) 60; 24, (b) $\binom{100}{50}(0{\cdot}24)^{50} \approx 1{\cdot}05\times10^{-62}$,

(c) $(0{\cdot}4)^{100} \approx 1{\cdot}607\times10^{-40}$, (d) $(0{\cdot}6)^{100} \approx 6{\cdot}533\times10^{-23}$]

[*Hint.* Use the brand share model of Section 1.6.]

*The cost that will make the company policy optimal.

APPENDIX A

SIMULATION

A.1 'To simulate' means 'to make similar'. Thus, simulation, in its wider sense, is synonymous with model-building. However, very often as we are doing here, the word simulation is used in its narrow sense to mean what more precisely should be called *Monte Carlo simulation*. The words Monte Carlo, derived from the name of a place known for its casinos, have come to signify use of probability, and Monte Carlo simulation means the method of generating values of a random variable from its probability distribution, as explained below.

A.2 Monte Carlo Simulation

Consider a random variable X with Probability Distribution Function (P.D.F.) $F(x)$, so that

(1) $$\mathrm{P}(X \leqslant x) = F(x)$$

and

(2) $$\mathrm{P}(x < X < x+\Delta x) = f(x)\Delta x$$

where $f(x) = \dfrac{d}{dx}F(x)$ is the corresponding probability density function (p.d.f.).

Take a random fraction $R \in [0, 1)$ by affixing a decimal point before a 2 or 3-digit random number, e.g. a number 23 may be changed to the fraction $R = {\cdot}23$ or $0{\cdot}23$. The fraction R is said to correspond to the value x of the variable X, given by

(3) $$F(x) = R$$

or,

(4) $$x = F^{-1}(R)$$

where F^{-1} is the inverse function of F.

The justification for the above method lies in the fact that the method implies that

(5) $$\mathrm{P}(x < X < x+\Delta x) = F(x+\Delta x) - F(x) \approx f(x)\Delta x$$

to the first order terms in Δx, which agrees with (2) and thus the value x is generated according to the given probability distribution.

Thus, by taking a number of random numbers we can have random fractions $R_1, R_2, \ldots$ and generate as many values $x_1, x_2, \ldots$ of X.

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The above method of simulation applies equally well to the continuous and discrete variables, the only difference being that in the latter case $F(x)$ is a step function and, therefore, to each value of the variable a set of random numbers can be associated. Thus, if the discrete variable takes the values x_i with probabilities p_i, $i = 1, 2, \ldots, n$, then

$$R \in (P_{i-1}, P_i] \Rightarrow X = x_i \tag{6}$$

where $P_i = \sum_{r=1}^{i} p_r$.

A.3 Need for Simulation

The method of simulation described above can be used to solve any problem concerning random variables, but the real advantage of simulation is obtained for solving problems where an analytical solution either does not exist or is too involved to be practicable. Whereas analytical methods enable us to express the outcome of a decision as a function of the decision variables and, therefore, help us in arriving at the optimal decision, for example, through the use of calculus or a technique like linear programming, simulation helps us to work out the effect of each feasible decision and then to choose the *best* decision out of those tried. Thus, simulation helps us in arriving at a *satisficing** rather than the *optimal* decision. As mentioned in Section A.2, we can use simulation to generate as many values of a random variable as we like by using sufficiently many random numbers. Evidently, by generating a number of values of the random variable or variables, we obtain a sample of values of the variables in question from the population of values corresponding to the given probability distribution. As is true for all sampling, the precision increases with the size of the sample. However, the question "how large the simulated sample should be" is an involved one, and the answer lies in striking a balance between the cost of sampling and the cost of making a wrong decision. Besides, the statistical principles pertaining to sampling theory, like *variance-reduction techniques*,** are applicable.

Ex. 1. Simulate, taking 20 consecutive arrivals, the M/M/1 queueing system with mean arrival rate of 10 per hour and mean service time of 4 minutes, and obtain the estimates for L_q, W_q, and the proportion of time that the server is idle.

For generating values of interarrival times we take 20 consecutive two-digit random numbers. Choosing the numbers from the 26th row of the

*See, for example, Grassmann (1981), Chap. 10.
**See, for example, Tocher (1963), p. 93.

'Table of Random Digits' in the book, we have the numbers 56, 49, 07, ..., 28 as given in Column 1 of Table 1 below.

Let us, for example, take the second number 49. Corresponding to this we have the value of interarrival time, x, given by

$$1 - e^{-\lambda x} = 0{\cdot}49$$

whence

$$x = -(ln\ 0{\cdot}51)/\lambda = -6\ ln\ 0{\cdot}51 \text{ min.} \approx 4{\cdot}04 \text{ min.}$$

This value is shown in Column 2 of Table 1. Similarly for other values of interarrival time shown in the table.

TABLE 1

R.N.*	Inter-Arrival Time (min.)	Arrival Time	R.N.*	Service Time (min.)	Service Begins	Service Ends	Waiting Time (min.)	Idle Time (min.)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
56	4.93	4.93	92	10.10	4.93	15.03	—	4.93
49	4.04	8.97	60	3.67	15.03	18.70	6.06	—
07	0.44	9.41	67	4.43	18.70	23.13	9.29	—
56	4.93	14.34	66	4.32	23.13	27.45	8.79	—
89	13.24	27.58	67	4.43	27.58	32.01	—	0.13
70	7.22	34.80	87	8.16	34.80	42.96	—	2.79
23	1.57	36.37	09	0.38	42.96	43.34	6.59	—
35	2.58	38.95	65	4.20	43.34	47.54	4.39	—
11	0.70	39.65	49	2.69	47.54	50.23	7.89	—
20	1.34	40.99	31	1.48	50.23	51.71	9.24	—
32	2.31	43.30	11	0.47	51.71	52.18	8.41	—
26	1.81	45.11	89	8.83	52.18	61.01	7.07	—
98	23.47	68.58	67	4.43	68.58	73.01	—	7.57
01	0.06	68.64	49	2.69	73.01	75.70	4.37	—
31	2.23	70.87	41	2.11	75.70	77.81	4.83	—
82	10.29	81.16	19	0.84	81.16	82.00	—	3.35
05	0.31	81.47	07	0.29	82.00	82.29	0.53	—
60	5.50	86.97	23	1.05	86.97	88.02	—	4.68
48	3.92	90.89	81	6.64	90.89	97.53	—	2.87
28	1.97	92.86	11	0.47	97.53	98.00	4.67	—
							82.13	26.32

* Random Number.

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Again, we read 20 two-digit random numbers for service time, say from the last two columns of the random digit table, starting with the 26th row,* and obtain : 92, 60, ..., 11 as shown in Col. 4. From these we obtain the service time values as shown in Col. 5. The remaining procedure is self-explanatory. Taking the time of the starting of the system as the time zero, we work out the arrival times and the times that service begins and ends.

Since the server is idle for 26·32 minutes out of the total time duration of 98·00 minutes covered above, we have

$$\text{Proportion of time the server is idle} = \frac{26{\cdot}32}{98{\cdot}00} \approx 0{\cdot}27$$

Again, the 20 customers wait for a total of 82·13 minutes. Hence, $W_q = 82{\cdot}13/20 \approx 4{\cdot}11$ min.

Now, the total 82·13 minutes of the column for waiting time can be considered to be in customer-minute units. So, by dividing this by the total time of 98·00 minutes, we have the average number of customers waiting (in queue), that is,

$$L_q = \frac{82{\cdot}13}{98} \approx 0{\cdot}84 \text{ customer}$$

Note. For the problem discussed above, analytical results exist and are well-known. The problem is given just for the sake of illustration. The analytical values are :

$$\text{P\{server is idle}\} = P_0 = 1-\rho = 1-\frac{10}{15} \approx 0{\cdot}33$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{4/9}{1/3} \approx 1{\cdot}33$$

$$W_q = L_q/\lambda = \frac{4}{3}\times 6 \text{ min.} = 8 \text{ min.}$$

Ex. 2. Work out Ex. 1 taking the mean service time as 2 minutes.

The calculation work is shown in Table 2, where the same sets of random numbers have been chosen for comparison's sake, though we could have

* The choice of the same 26-th row for generating arrival and service times is just coincidental. The two starting points can be taken anywhere and do not have to be related to each other *in any way*.

chosen different sets of random numbers. Now, the results obtained through simulation are seen to be

$$\text{P\{server is idle}\} = \frac{58{\cdot}59}{94{\cdot}44} \approx 0{\cdot}62$$

$$L_q = \frac{13{\cdot}99}{94{\cdot}44} \approx 0{\cdot}15$$

$$W_q = \frac{13{\cdot}99}{20} \approx 0{\cdot}70 \text{ min.}$$

In the present case, $\rho = 10/30 = 1/3$; so the analytical results are

$$P_0 = 1-\rho = 2/3 \approx 0{\cdot}67$$

$$L_q = \frac{\rho^2}{1-\rho} = \frac{1/9}{2/3} = \frac{1}{6} \approx 0{\cdot}167 \text{ customer}$$

$$W_q = L_q/\lambda = (1/6)\times 6 \text{ min.} = 1 \text{ min.}$$

TABLE 2

R.N.* (1)	Inter-Arrival Time (min.) (2)	Arrival Time (3)	R.N.* (4)	Service Time (min.) (5)	Service Begins (6)	Service Ends (7)	Waiting Time (min.) (8)	Idle Time (min.) (9)
56	4.93	4.93	92	5.05	4.93	9.98	—	4.93
49	4.04	8.97	60	1.83	9.98	11.81	1.01	—
07	0.44	9.41	67	2.22	11.81	14.03	2.40	—
56	4.93	14.34	66	2.16	14.34	16.50	—	0.31
89	13.24	27.58	67	2.22	27.58	29.80	—	11.08
70	7.22	34.80	87	4.08	34.80	38.88	—	5.00
23	1.57	36.37	09	0.19	38.88	39.07	2.51	—
35	2.58	38.95	65	2.10	39.07	41.17	0.12	—
11	0.70	39.65	49	1.35	41.17	42.52	1.52	—
20	1.34	40.99	31	0.74	42.52	43.26	1.53	—
32	2.31	43.30	11	0.23	43.30	43.53	—	0.04
26	1.81	45.11	89	4.41	45.11	49.52	—	1.58
98	23.47	68.58	67	2.22	68.58	70.80	—	19.06
01	0.06	68.64	49	1.35	70.80	72.15	2.16	—
31	2.23	70.87	41	1.06	72.15	73.21	1.28	—
82	10.29	81.16	19	0.42	81.16	81.58	—	7.95
05	0.31	81.47	07	0.15	81.58	81.73	0.11	—
60	5.50	86.97	23	0.52	86.97	87.49	—	5.24
48	3.92	90.89	81	3.32	90.89	94.21	—	3.40
28	1.97	92.86	11	0.23	94.21	94.44	1.35	—
							13.99	58.59

* Random Number.

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We observe that the simulation results are quite close to the analytical ones in Ex. 2, while they varied considerably in Ex. 1. Generally speaking, we can expect better match for small values of ρ. We know that for $\rho \geqslant 1$, the steady state does not exist; consequently for ρ close to 1 we should expect a long time to elapse before the steady state is obtained. For better results, it is advisable that several initial values are discarded and simulation estimates are based on the last few. For example, one may do the simulation exercise for 30 arrivals and base the calculations on, say, the last 10 of these.

APPENDIX B

PROBABILITY AND TRANSFORMS

B.1 Expectations and Moments

Let X be a continuous random variable (r.v.) with (cumulative) Probability Distribution Function (P.D.F.) $F(x)$, so that*

(1) $$\mathrm{P}\{X \leqslant x\} = F(x)$$

The corresponding probability density function (p.d.f.) is $f(x)$, given by

(2) $$\mathrm{P}\{x < X < x+dx\} = f(x)dx$$

It is easy to see from calculus that $F(x)$ and $f(x)$ are related as follows :

(3) $$f(x) = \frac{d}{dx}F(x)$$

and

(4) $$F(x) = \int_{-\infty}^{x} f(u)du$$

We define the *expectation*, or *expected value*, of a function, $g(X)$, of X to be

(5) $$E[g(X)] = \int_R g(x)dF(x) = \int_R g(x)f(x)dx$$

the integrals being taken over the range R of X. Most of the r.v.'s considered by us are non-negative and extend over the interval $[0, \infty)$. Moreover, out of the two equivalent forms given by the second and the third members of (5)**, we shall usually prefer the latter, so that we can write

(6) $$E[g(X)] = \int_0^\infty g(x)f(x)dx\dagger$$

Let a be any real number (which is usually taken to belong to the range R of X) and r an integer $\geqslant 1$; then

(7) $$E[(X-a)^r] = \int_0^\infty (x-a)^r f(x)dx$$

---

* 'P(A)' stands for 'probability that the event A occurs'.

** The second member expresses expectation as a *Stieltjes* or *Riemann-Stieltjes* integral; see for example, Chaudhry and Templeton (1983), pp. 345–348.

† The Steiltjes integral form is more general, since it includes both the cases when X is a *continuous* or a *discrete* r.v. (in the latter case $F(x)$ is a step function and so, at some points, $f(x)$ cannot be defined). Still, for the sake of simplicity we use the other (Riemann integral) form, with which the reader is expected to be more familiar.

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is called the r-th moment, about the value a, of X (or of the distribution of X). If $a = 0$, we get the r-th moment about the origin, usually denoted by μ_r'; thus

$$\mu_r' = E[X^r] = \int_0^\infty x^r f(x)dx \tag{8}$$

is the r-th moment of X about the origin. In particular, the first moment of X about the origin :

$$\mu_1' = E[X] = \int_0^\infty xf(x)dx \tag{9}$$

is called the *mean* of X and is often denoted by $\bar{x}$. From the foregoing it is clear that the mean is the *expected value* of the r.v. X, and corresponds to the notion of an ordinary (arithmetic) *average*. Sometimes, though less technically, the words *mean* and *average* are used synonymously. Moments about the mean, $\bar{x}$, play an important part in statistics and probability. The r-th moment about the mean is usually denoted by μ_r; thus

$$\mu_r = E[(X-\bar{x})^r] = \int_0^\infty (x-\bar{x})^r f(x)dx \tag{10}$$

The second moment about the mean, μ_2, is called the *variance* of X. The variance, and its positive square-root known as the standard deviation (S.D.) and denoted by σ_X, are measures of *dispersion* or spread of the values of the r.v. X around the *central* value denoted by the mean $\bar{x}$. The variance is given by

$$\text{Var}[X] = \mu_2 = E[(X-\bar{x})^2] = \sigma_X^2 \tag{11}$$

It is easy to show that

$$\mu_2 = \mu_2' - \mu_1'^2 = E[X^2] - \bar{x}^2 \tag{12}$$

Further, it can be seen from the definition (11) that the variance, and hence the standard deviation, will vanish iff (if and only if) we have a *deterministic* case when the variable can have only one value, say,

$$f(x) = \delta(x-\bar{x}) \tag{13}$$

where $\delta(\cdot)$ is the Dirac delta function.

The mean and the variance (or S.D.) together provide information which is usually sufficient for most of application studies. The mean alone gives a central value and, without the knowledge of variability around this value, often provides too little information. Moments higher than the second provide still more information about the distribution and shape of the p.d.f. curve. For example, *skewness*, a measure of asymmetry, is given by

$$\text{skewness}^* = \mu_3/\mu_2^{3/2} = \mu_3/\sigma^3 \tag{14}$$

and *kurtosis*, a measure of peakedness of the p.d.f. curve, is

$$\text{kurtosis} = \mu_4/\mu_2^2 = \mu_4/\sigma^4$$

Discrete Variables

The foregoing discussion pertains primarily to a continuous variable. With slight change of notation there are corresponding results and theory for discrete random variables. Let N be a discrete r.v., taking on values 0, 1, 2, ... with

$$\mathrm{P}\{N = n\} = p_n, \quad n = 0, 1, 2, \ldots \tag{16}$$

Then, the mean, or the expected value, of N is

$$E[N] = \sum_{n=0}^\infty np_n = \bar{n}, \text{ say} \tag{17}$$

The r-th moment about the mean is

$$\mu_r = E[(N-\bar{n})^r] = \sum_{n=0}^\infty (n-\bar{n})^r p_n \tag{18}$$

and the r-th moment about the origin is

$$\mu_r' = E[N^r] = \sum_{n=0}^\infty n^r p_n \tag{19}$$

And, in particular, the variance is

$$\begin{aligned}\mu_2 = \text{Var}[N] = E[(N-\bar{n})^2] &= \sum_{n=0}^\infty (n-\bar{n})^2 p_n \\ &= \sum_{n=0}^\infty n^2 p_n - \bar{n}^2 \\ &= \mu_2' - \mu_1'^2\end{aligned} \tag{20}$$

B.2 The Laplace Transform

Given any function $f(t)$, of a real variable t, the Laplace transform (L.T.) of $f(t)$ is defined as

$$\mathscr{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt \tag{1}$$

where s is a complex variable such that, for the sake of convergence, $\text{Re}(s) > 0$†, where $\text{Re}(s)$ signifies 'the real part of s'. Thus $\mathscr{L}[f(t)]$ is a function of s and

* Alternatively, some authors define skewness $= \mu_3^2/\mu_2^3$.

† In most cases discussed by us, the integral converges for $\text{Re}(s) > 0_+$ (Cf. (6)–(7), following.)

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is usually denoted by $\bar{f}(s)$, $f^*(s)$, and sometimes by corresponding lower case letters, as

$$\mathcal{L}[F(t)] = f(s)$$

From (1) and B.1(6) it is clear that if $f(x)$ be the p.d.f. of a random variable $X \in [0, \infty)$, we have

$$\mathcal{L}[f(t)] = E[e^{-sX}] \tag{2}$$

Sometimes, the Laplace-Stieltjes transforms (L.S.T.) are used. The L.S.T. of a function, say $g(t)$, may be defined as

$$\mathcal{LS}[g(t)] = \int_0^\infty e^{-st} dg(t), \ \mathrm{Re}(s) > 0 \tag{3}$$

There seems to be no standard notation for L.S.T., though some authors use (**) as

$$\mathcal{LS}[g(t)] = g^{**}(s) \tag{4}$$

Very often we use the L.T. of a probability density function (p.d.f.) or the L.S.T. of a Probability Distribution Function. Let $F(\cdot)$ be the P.D.F. of a random variable X, defined by B.1(1), and $f(\cdot)$ the corresponding p.d.f., defined by B.1(2); then by definition it is clear that

$$\mathcal{L}[f(t)] = \mathcal{LS}[F(t)] \tag{5}$$

and we shall denote both of them by $\bar{f}(s)$. Thus,

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-st} dF(t), \ \mathrm{Re}(s) \geqslant 0 \tag{6}$$

It may be noted that

$$\bar{f}(0) = 1 \tag{7}$$

since the left-hand member of (7) stands for the total probability of the r.v. X.

Some Properties of the Laplace Transform

Let $f(t)$ be any function, *not necessarily* a p.d.f. (though in most applications discussed by us, it *is* a p.d.f.) and $\bar{f}(s)$ its L.T.; then, since the L.T. is defined as an integral, all properties of integrals become applicable to it. In particular, we have

$$\mathcal{L}[cf(t)] = c\mathcal{L}[f(t)] = c\bar{f}(s) \tag{8}$$

where c is constant,

$$\mathcal{L}[f(t)+g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] = \bar{f}(s) + \bar{g}(s) \tag{9}$$

Also, using the definition of the L.T. and integrating by parts, we can show that

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \mathcal{L}[f'(t)] = s\bar{f}(s) - f(0) \tag{10}$$

provided $\lim_{t \to \infty} [e^{-st} f(t)] = 0$. [This condition is satisfied in most applications.]

Similarly, using the definition and changing the order of integration in the resulting double integral, we have

$$\mathcal{L}\left[\int_0^t f(u) du\right] = \bar{f}(s)/s \tag{11}$$

From (11), it follows that if $F(\cdot)$ is a P.D.F., we have

$$\mathcal{L}F(t) = \bar{f}(s)/s = \mathcal{L}[f(t)]/s \tag{12}$$

Also, if a is a constant, we have

$$\mathcal{L}[e^{-at} f(t)] = \bar{f}(s+a) \tag{13}$$

which is evident from definition.

And, from (6) on differentiating n times w.r.t. s, we have

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} (\bar{f}(s)), \quad (n \text{ a positive integer}) \tag{14}$$

An important Laplace transform very often used is

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st} t^n dt \tag{15}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n du, \quad \text{on using the substitution } t = \frac{u}{s}$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, \quad \text{using the definition of the Gamma function}$$

$$= \frac{n!}{s^{n+1}}, \quad \text{if } n \text{ is a positive integer}$$

From (14), on setting $s = 0$ on both the sides, we have the important result that for a random variable with p.d.f. $f(\cdot)$, the n-th moment about the origin is given by

$$\mu_n' = (-1)^n \bar{f}^{(n)}(0) \tag{16}$$

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where $\bar{f}^{(n)}(0)$ signifies the n-th derivative of $\bar{f}(s)$ at $s = 0$. As a special case of (16) the mean is given by

$$\mu_1' = -\bar{f}'(0) \tag{17}$$

B.3 Convolutions (Continuous Variables)

Let $f(t)$ and $g(t)$ be two functions of a (non-negative) varibable t; then the convolution of $f(t)$ and $g(t)$, denoted by $f(t) * g(t)$, is defined as follows:

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du \tag{1}$$

$$= \int_0^t f(t-u)g(u)du, \quad \text{as is evident on substitution}$$

$$= g(t) * f(t).$$

The convolution of three functions $f_i(t)$, $i = 1, 2, 3$, is defined as

$$f_1(t) * f_2(t) * f_3(t) = f_1(t) * (f_2(t) * f_3(t)) \tag{2}$$

From the definition, it is clear that convolution (*) regarded as a binary operation on the set of functions of t, possesses both the commutative and associative properties, so that in the right-hand member of (2) the functions $f_i(t)$, $i = 1, 2, 3$, can be permuted in any of the 6 different ways.

The definition (2) can obviously be extended to the case of n functions $f_i(t)$, $i = 1, \ldots, n$, for any integer n. Consider the case when

$$f_1(t) = f_2(t) = \ldots = f_n(t) = f(t) \tag{3}$$

Then, the convolution of the functions $f_i(t)$, $i = 1, \ldots, n$, becomes

$$f(t) * f(t) * \ldots \text{(repeated } n \text{ times)} = f(t)^{*(n)} \tag{4}$$

which is called the n-fold convolution of $f(t)$ with itself. In particular, the (2-fold) convolution of $f(t)$ with itself is, from (13),

$$f(t) * f(t) = \int_0^t f(u)f(t-u)du \tag{5}$$

the epithet '2-fold,' in fact, being redundant.

The Convolution Theorem of Laplace Transforms

Let $h(t)$ be the convolution of two functions, $f(t)$ and $g(t)$, of t, which for our application purposes may be taken to be a non-negative real variable in $[0, \infty)$; then the L.T. of $h(t)$ equals the product of the L.T.'s of $f(t)$ and $g(t)$, that is, with the notation introduced above,

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] \tag{6}$$

or, written alternatively,

$$\bar{h}(s) = \bar{f}(s)\bar{g}(s) \tag{7}$$

The proof follows from the definitions of L.T. and convolution on changing the order of integration in the resulting double integral.

(6) has obvious generalization to the case of n functions (for any positive integer n). In particular, we have

$$\mathcal{L}[f(t)^{*(n)}] = (\mathcal{L}[f(t)])^n = (\bar{f}(s))^n \tag{8}$$

Convolution and Probability

Let X, Y be two *independent* (non-negative) random variables having probability density functions (p.d.f.) $f(x)$ and $g(y)$ and let $Z = X + Y$. Then, using the multiplication theorem of probabilities for independent r.v.'s, we see that the density function for Z, say $h(z)$, is given by

$$h(z) = \int_0^z f(z)g(z-x)dx = f(z) * g(z) \tag{9}$$

and hence, using (6), the L.T. of $h(z)$ is given by

$$\bar{h}(s) = \bar{f}(s)\bar{g}(s) \tag{10}$$

where $\bar{f}(\cdot)$, $g(\cdot)$ are the L.T.'s of $f(\cdot)$, $g(\cdot)$, respectively.

Like (6), (10) is also capable of generalization to the case of any number of independent random variables. In particular, if $X_1, X_2, \ldots, X_n$ be n i.i.d.r.v. (independent *identically* distributed random variables) having the same p.d.f., $f(x)$, the L.T. of the p.d.f. $h(z)$, say, of their sum

$$Z = \sum_{i=1}^{n} X_i \tag{11}$$

is given by

$$\bar{h}(s) = (\bar{f}(s))^n \tag{12}$$

where

$$\bar{f}(s) = \int_0^\infty e^{-sx} f(x)dx \tag{13}$$

is the L.T. of $f(x)$.

(12) is very often used in queueing theory; e.g. Cf. 1.4(19), 2.3(11). For an example of the use of (10) one may refer to 3.3(34).

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It may be noted that (10) and (12) will hold if we replace the p.d.f. by P.D.F. (probability *distribution* function) and the L.T. by L.S.T. and use B.2(5). Note further that the analogue of (9) in terms of the distribution functions will be

$$H(z) = \int_0^z G(z-x)dF(x) = F(z) * G(z) \tag{14}$$

where $F(\cdot)$, $G(\cdot)$ and $H(\cdot)$ denote P.D.F. of X, Y, Z, respectively, and $\tilde{f}(s)$, etc. in (10), (12) are defined as in B.2(6).

B.4 Convolutions (Discrete Variables)

Let p_n, q_n $(n = 0, 1, \ldots)$ be two functions of a discrete variable; then

$$p_n * q_n = \sum_{m=0}^{n} p_m q_{n-m} \tag{1}$$

is called the convolution of p_n and q_n.

As in the case of functions of continuous variables, the above definition admits of similar extensions to the definition of convolutions of more than two functions and convolution of a function with itself. (See B.3 (2), (4).)

Generating Functions

Given a function p_n $(n = 0, 1, \ldots)$, the function

$$\mathrm{P}(z) = \sum_{n=0}^{\infty} p_n z^n \tag{2}$$

is called the generating function† of p_n. The variable z may be regarded as a complex variable with a condition imposed on z to ensure convergence of $\mathrm{P}(z)$.

If p_n be the probability mass function (p.m.f.) of a discrete random variable N, say, i.e.

$$\mathrm{P}(N = n) = p_n \tag{3}$$

we shall require $|z| \leqslant 1$ for convergence. Also, in this case $\mathrm{P}(z)$, being the generating function of the p.m.f. of N, is called the p.g.f. of N.

Using the discrete analogue of B.1(6), we see that

$$\mathrm{P}(z) = E[z^N] \tag{4}$$

†The generating function is often called the Z-transform, and may be taken as the discrete analogue of the Laplace transform. However, some authors define the Z-transform of p_n as $\sum_{n=0}^{\infty} p_n z^{-n}$.

Let p_n, q_n be the probability mass functions (p.m.f.) of two *independent* discrete random variables N_1, N_2, respectively, i.e.,

$$\mathrm{P}(N_1 = n) = p_n \quad (n = 0, 1, \ldots) \tag{5}$$

$$\mathrm{P}(N_2 = n) = q_n \quad (n = 0, 1, \ldots). \tag{6}$$

Then, using the multiplication theorem of probability we see that the p.m.f. of the sum N_1+N_2 is

$$\mathrm{P}(N_1+N_2 = n) = \sum_{m=0}^{n} p_m q_{n-m} = p_n * q_n = r_n, \text{ say} \tag{7}$$

Also, the generating function of r_n, i.e., the p.g.f. of N_1+N_2 is

$$\begin{aligned} R(z) = \sum_{n=0}^{\infty} r_n z^n &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} p_m q_{n-m} z^n \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} p_m q_{n-m} z^n \text{ (on changing the order of summation)} \\ &= \sum_{m=0}^{\infty} p_m z^m \sum_{n=m}^{\infty} q_{n-m} z^{n-m} \\ &= \mathrm{P}(z)Q(z) \end{aligned} \tag{8}$$

(8) is the discrete analogue of B.3(10) and can likewise be extended to the sum of any number of discrete random variables. In particular, if $N_1, N_2, \ldots, N_r$ be i.i.d. r.v.'s having the same p.g.f. $\mathrm{P}(z)$, the p.g.f. $R(z)$ of $N_1+N_2+\ldots+N_r$ will be given by

$$R(z) = (\mathrm{P}(z))^r \tag{9}$$

Some Properties of Z-Transforms

Let us write the Z-transform (or generating function) of p_n as $Z[p_n]$, i.e.,

$$Z[p_n] = \sum_{n=0}^{\infty} p_n z^n \tag{10}$$

Thus, $Z[p_n] = \mathrm{P}(z)$, as given by (2). Then, the following relations are evident from definition

$$Z[cp_n] = cZ[p_n] \tag{11}$$

$$Z[p_n+q_n] = Z[p_n]+Z[q_n] \tag{12}$$

And, (8) can be written as

$$Z[p_n * q_n] = Z[p_n]Z[q_n]$$

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Also, we have

(14)
$$Z[p_{n+r}] = \sum_{n=0}^{\infty} p_{n+r} z^n$$
$$= z^{-r} \sum_{n=0}^{\infty} p_{n+r} z^{n+r}$$
$$= z^{-r} \left\{ Z[p_n] - \sum_{n=0}^{r-1} p_n z^n \right\}$$

APPENDIX C

MISCELLANEOUS TOPICS

C.1 Rouché's Theorem

A function $f(z)$ of a complex variable z is said to be *analytic* in a region R if it is single-valued and differentiable at every point of R.

If two functions $f(z)$ and $g(z)$ are analytic inside and on a closed contour C on the complex z plane *and* $|g(z)| < |f(z)|$ *on* C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

C.2 Lagrange's Differential Equation

The general solution of the partial differential equation

(1)
$$P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R$$

where P, Q, R are functions of x, y, z, is

(2)
$$\phi(u, v) = 0$$

or, equivalently,

(3)
$$v = f(u)$$

where ϕ, f are arbitrary functions, determined from initial conditions, and $u = a$, $v = b$, a and b being arbitrary constants, are two independent solutions of the corresponding simultaneous *ordinary* differential equations

(4)
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(4) are known as the Lagrange system of equations.

C.3 Difference Equations

Just as a consideration of increments in the argument(s) of a function of continuous variable(s) gives rise to differential equations, a similar consideration in the case of a function of discrete variable(s) yields equations known as *difference equations*. An example of a homogeneous linear difference equation of order n with constant coefficients is

(5)
$$a_0 y_{x+n} + a_1 y_{x+n-1} + \ldots + a_n y_x = 0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants.

In order to solve (5), we use an operator E, defined by

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(6) $$E^r y_x = y_{x+r},\ r = 0, 1, 2, \ldots$$

so that operation with E results in enhancing the argument x by unity. Using E, (5) can be written as

(7) $$a_0 E^n y_x + a_1 E^{n-1} y_x + \ldots + a_n y_x = 0$$

or, separating the operator,

(8) $$(a_0 E^n + a_1 E^{n-1} + \ldots + a_n) y_x = 0$$

The equation

(9) $$a_0 E^n + a_1 E^{n-1} + \ldots + a_n = 0$$

regarded as an equation (of degree n) in E is called the *characteristic equation*, or *determinantal equation*, of (5). Let (9) have the n distinct roots

$$E = m_1, m_2, \ldots, m_n$$

Then, the general solution of (5) is

(10) $$y_x = c_1 m_1^x + c_2 m_2^x + \ldots + c_n m_n^x$$

where $c_1, \ldots, c_n$ are arbitrary constants, to be determined from initial conditions.

If the roots of (9) are not all distinct, let $m_2 = m_1$, say, the other roots being all distinct and not equal to m_1; then the first two terms of (10), corresponding to the equal roots m_1, m_2, will be replaced by $(c_1 + c_2 x) m_1^x$. This result has an obvious generalization to a root repeated any number of times.

The reader will have observed the analogy between the above and the corresponding theory regarding differential equations.

For a more detailed but introductory reading on difference equations, the reader may refer to Yamane (1968).

C.4 Differential-Difference Equations

A consideration of variations in a continuous variable often results in a differential equation and, in the same way, a consideration of variations in a discrete variable leads to a difference equation. In queueing theory, when we deal with functions like $P_n(t)$, where n is a discrete variable and t a continuous one, and consider changes in both n and t, the result is a differential-difference equation (sometimes also called a difference-differential equation). Examples of this are 1.5(4), (6). By using the Z-transform or generating function, the 'difference' aspect of the equation is done away with, thus yielding a 'pure' differential equation, see, for example, 2.2(4). However,

if we use Laplace transforms, the 'differential' aspect is got rid of and we are left with a difference equation, as, for example, in Srivastava and Kashyap (1982), Eq. 1.1(2). These resulting differential or difference equations may be solved by usual methods*. However, usually both the Z and the Laplace transforms are used thus reducing the equation to a simple *algebraic equation*, as is done in 2.2(8), 2.3(7). It may be noted that consideration of the steady state of the system makes the derivatives w.r.t. the time t vanish, thus doing away with the 'differential' aspect of the equation, as in 1.5(7), (8). As a further illustration, refer to Section 3.6, where, since we consider the steady state of the system from the very beginning, we obtain a differential-difference equation in $P_n(x)$, given by 3.6(9), (10). Otherwise, consideration of the *transient* state would have resulted in what may be called a *partial-differential-difference* equation, in $P_n(x, t)$ as, for example, in Srivastava and Kashyap (1982), Eq. 4.4(3)–(5). For more details on differential-difference equations, one may refer to Bellman and Cooke (1963).

Thus, the use of the Laplace and/or the Z-transforms results in a considerable simplification of the queue equations, but as is true for all 'benefits', this simplification, too, can be had at a certain 'cost'. The cost in this case is that we obtain solutions in terms of transforms, which should be *inverted* to arrive at a transform-free, or *explicit*, solution. This necessitates consideration of the problem of inversion of the transforms.

C.5 Inversion of Laplace Transforms

If, as heretofore,

(1) $$\mathscr{L}[f(t)] = \bar{f}(s)$$

we say that $f(t)$ is the *inverse Laplace transform* or *Laplace-inverse* of $\bar{f}(s)$, and write

(2) $$\mathscr{L}^{-1}[\bar{f}(s)] = f(t)$$

A simple way of finding inverse Laplace transforms is by using the results for Laplace transforms in the reverse order. Thus since, for example, we have

(3) $$\mathscr{L}[e^{-at}] = 1/(s+a)$$

we can write

(4) $$\mathscr{L}^{-1}[1/(s+a)] = e^{-at}$$

* The method of solution of a differential equation is simple and is covered in undergraduate calculus. The method for a difference equation is briefly discussed in Section C.3.

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The general method of finding the inverse Laplace transform is through the use of the complex inversion formula of Laplace transforms which states that if $f(t)$ and $\bar{f}(s)$ are related as in (1), we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)ds, \quad t > 0 \tag{5}$$

$$f(t) = 0, \quad t < 0 \tag{6}$$

where the integration in (5) is performed along the line $z = c$ in the complex z-plane ($z = x+iy$), the real number c being chosen arbitrarily subject to the condition that $z = c$ lies to the right of all singularities* of $\bar{f}(s)$.

For more details regarding the complex inversion formula, one may refer to Kleinrock (1975) or to books on Laplace transforms like Widder (1946) and Doetsch (1974).

C.6 Inversion of Z-Transforms

This, too, can be achieved by using the Z-transform results in the reverse order as in the case of the L.T. However, the inversion of a Z-transform

$$Z[\mathrm{P}_n] = \sum_{n=0}^{\infty} \mathrm{P}_n z^n \tag{1}$$

implies that we have to evaluate P_n when the function $Z[\cdot]$ in (1) is given. Thus the problem is simply that of writing down the coefficient of z^n in a function of z, for which several methods are known to the average reader. The basic problem is that of expansion of $Z[\cdot]$ in powers of z. In many cases, use of known expansions like the *binomial theorem* is found useful; see, for example, Eq. 1.7(10), (11). Sometimes, $Z[\cdot]$ may have to be resolved into partial fractions before the use of the binomial theorem. Also, in a complicated equation, inversion may be effected by equating coefficients of like powers of z on both sides; for an example see Problem Set 4, Problem 5.

For a discussion of the general method of inversion of $Z[\cdot]$ via the use of complex variable theory, the reader may refer, for example, to Kleinrock (1975), and Chaudhry and Templeton (1983).

For an interesting example of inversion of generating functions, one may refer to Problem Set 4, Problem 12; one may also refer to Chaudhry *et al.* (1987) and Brière (1987) for some more complex examples of inversion of generating functions using roots of characteristic equations in connection with bulk queues.

* A *singular point*, of a function of a complex variable, is a point where the function fails to be analytic.' Singularities' include singular points, and *essential* singular points. (Cf. Churchill (1972)).

C.7 Differentiation of a Definite Integral

Consider the definite integral

$$\phi(x) = \int_a^b f(x, z)dz \tag{1}$$

where the limits of integration a, b are functions of x.

It can be shown by using the definition of the derivative that

$$\frac{d\phi(x)}{dx} = \int_a^b \frac{\partial f(x,z)}{\partial x}dz + \frac{db}{dx}f(x,b) - \frac{da}{dx}f(x,a) \tag{2}$$

In the special case when a, b are constants, the last two terms in (2) will vanish.

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APPENDIX D

LIST OF SYMBOLS

The following is the list of symbols and notation most commonly used in the text. These and other symbols and notation are, however, also explained in the text when they are introduced.

D.1 *Notation for Queueing Systems*

A queueing system is generally denoted by

$$A^X/B^Y/c/N$$

where A and B, respectively, stand for the type of inter-arrival time and service time distributions, c is the number of channels (servers) and N is the maximum number of customers admissible to the system. The superscripts X and Y, respectively, indicate that a variable number of customers may be arriving or may be served together in a group. When N is missing in the above notation, it means that N is infinite. A and B may be replaced by the following symbols with their meanings shown against each.

D	constant inter-arrival or service times
E_k	k-Erlang distribution, i.e. the k-fold convolution of the exponential distribution
G	general (arbitrary) distribution, commonly used for service times (replacing B)
GI	general (arbitrary) independent input distribution, used for inter-arrival times (replacing A) in preference over 'G'
HE	hyper-exponential distribution
M	(negative) exponential distribution

Replacement of X by R indicates that the arriving group size is constant and equal to R.

D.2 *Symbols from Roman Alphabet*

$A(u)$	P.D.F. of inter-arrival times
$A(z)$	p.g.f. of arrival-group size ($= \sum_r a_r z^r$)

$\bar{a}$	mean arrival-group size ($= \sum_r r a_r$)
a_r	probability (mass) function of arrival-group size X, i.e., $P(X = r)$
$a(u)$, $\tilde{a}(s)$	p.d.f. of inter-arrival times and its L.T.
$B(v)$	P.D.F. of service times
$b(v)$, $\tilde{b}(s)$	p.d.f. of service times and its L.T.
E	finite difference operator as in $E^r P_n = P_{n+r}$
E[·], E(·)	expectation, or expected value or mean value, of
exp(x)	e^x where $e = 2{\cdot}71828 \ldots$ is the base of natural logarithms
FCFS	first-come, first-served
FIFO	first-in, first-out
$f(w)$, $\tilde{f}(s)$	p.d.f. of system time (waiting time in the system) and its L.T.
$f_q(w)$, $\tilde{f}_q(s)$	p.d.f. of queueing time (waiting time in the queue) and its L.T.
$I_\nu(\cdot)$	modified Bessel function of the first kind and order ν
i.i.d.	independently and identically distributed
$K(z)$	generating function of k_r
k_r	probability of r arrivals (service completions) during a service time (an inter-arrival time) of arbitrary duration
L	mean number of customers in system
L_q	mean number of customers in queue
L_q^+	mean number of customers in queue just after a departure
$\mathscr{L}(\cdot)$	Laplace transform of
$\mathscr{L}^{-1}(\cdot)$	inverse Laplace transform of
LCFS	last-come, first-served
LIFO	last-in, first-out
L.S.T.	Laplace-Stieltjes transform
L.T.	Laplace transform
N, n	number of customers in system
N_n	number in system just after the n-th departure (just before the n-th arrival)
N_q	number in queue
N_{serv}	number being served
$N(t)$	number in system at time t

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$O(x)$ †terms of the order of x as $x \to a$, so that
$$|O(x)| < Kx$$
for x close to a, K being a constant

$o(x)$ †terms negligible in comparison with x as $x \to a$, so that
$$\lim_{x \to a} \frac{o(x)}{x} = 0$$

$P(A)$ probability of the event A

$P(A \mid B)$ (conditional) probability of the event A, given that the event B has occurred

P_n (steady-state) probability that there are n customers in system

P_n^+ (steady-state) probability of finding n in system just after a departure

$P_n(t)$ probability that there are n in system at time t

$P(z)$ generating function of P_n $(= \sum_n P_n z^n)$

$P^+(z)$ generating function of P_n^+

p_{ij} one-step transition probability of going from state i to state j

$p_{ij}^{(n)}$ n-step transition probability of going from state i to state j

P.D.F. (cumulative) probability distribution function

p.d.f. probability density function

p.m.f. probability mass function

$Re(\cdot)$ real part of (a complex number)

r.v. random variable

s Laplace transform parameter

SIRO service in random order

t time

$t_1, t_2, \ldots$ times of successive arrivals (departures)

$u_1, u_2, \ldots$ inter-arrival times

$\mathrm{Var}[\cdot]$ variance of

$v_1, v_2, \ldots$ service times

W mean (waiting) time in system

W_q mean (waiting) time in queue or mean queueing time

† In the text, $a = 0$.

w, w_q times spent in system and in queue

z generating function parameter

$|z|$ modulus of z

D.3 *Greek-Alphabet and Miscellaneous Symbols*

$\Gamma(\cdot)$ gamma function, defined as
$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$
$$= (n-1)! \quad \text{if } n \text{ is an integer}$$

δ_{mn} Kronecker delta (= 1 if $m = n$; zero otherwise)

$\delta(x)$ Dirac delta function (= 0 except when $x = 0$, such that
$$\int_a^b \delta(x)dx = 1, \quad \text{if } a \leqslant 0 \leqslant b; \text{ zero otherwise}$$
Also, $\int_a^b \delta(x) f(x) dx = f(0)$, if $a \leqslant 0 \leqslant b$)

λ mean arrival rate

μ mean service rate

λ_n, μ_n mean arrival and service rates when there are n in system

μ_r, μ_r' r-*th* moment of a random variable about the mean and about the origin ($\mu_1' =$ mean, $\mu_2 =$ variance)

ρ (relative) traffic intensity, also called utilization factor, defined as (mean arrival rate to the system)/(mean service rate of the system). It is a pure number, but is taken to be measured in Erlangs.

σ^2 variance

σ_a^2, σ_v^2 variance of arrival-group size and service time respectively

$\sim$ †behaves as; $f(x) \sim g(x)$ as $x \to b$ if and only if
$$\lim_{x \to b} [f(x)/g(x)] = 1$$

$', ''$ primes (dashes) denoting differentiation

† Usually b is zero or infinite. In the text, b is infinite.

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APPENDIX E

SHORT TABLE OF LAPLACE TRANSFORMS

We give below some important results in Laplace transforms, most of which are used in the text. This table of Laplace transforms can also be used as one for inverse-Laplace transforms by reading entries from the second column and their inverses from the first column.

The function $f(t)$	$\mathcal{L}[f(t)] = \bar{f}(s)$
(1) $f(t)$	$\int_0^\infty e^{-st} f(t)\,dt$
(2) $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)\,ds$	$\bar{f}(s)$
(3) $af(t)+bg(t)$, a, b constants	$a\bar{f}(s)+b\bar{g}(s)$
(4) $f'(t)$	$s\bar{f}(s)-f(0)$ *
(5) $f^{(n)}(t)$, the n-th derivative of $f(t)$	$s^n\bar{f}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)$
(6) $\int_0^t f(u)\,du$	$\bar{f}(s)/s$
(7) $f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$	$\bar{f}(s)\bar{g}(s)$
(8) $e^{at}f(t)$	$\bar{f}(s-a)$
(9) $t^n f(t)$, $n \geqslant 0$	$(-1)^n\bar{f}^{(n)}(s) = \left(-\frac{d}{ds}\right)^n \bar{f}(s)$
(10) $f(t)/t$	$\int_s^\infty \bar{f}(u)\,du$
(11) 1	$1/s$
(12) t	$1/s^2$
(13) t^a, $n \geqslant 0$	$\Gamma(n+1)/s^{n+1}$ (= $n!/s^{n+1}$, if n is a positive integer)
(14) e^{at}	$1/(s-a)$
(15) $\delta(t)$, Dirac delta function	1
(16) $I_\nu(at)$, $\mathrm{Re}(\nu) > -1$	$a^\nu(s^2-a^2)^{-1/2}(s+\sqrt{s^2-a^2})^{-\nu}$, $\mathrm{Re}(s) > \lvert\mathrm{Re}(a)\rvert$
(17) $tI_\nu(at)$, $\mathrm{Re}(\nu) > -2$	$a^\nu(s+\nu\sqrt{s^2-a^2})(s^2-a^2)^{-3/2}\cdot(s+\sqrt{s^2-a^2})^{-\nu}$, $\mathrm{Re}(s) > \lvert\mathrm{Re}(a)\rvert$
(18) $t^{-1}I_\nu(at)$, $\mathrm{Re}(\nu) > 0$	$\nu^{-1}a^\nu(s+\sqrt{s^2-a^2})^{-\nu}$, $\mathrm{Re}(s) > \lvert\mathrm{Re}(a)\rvert$

APPENDIX F

TABLE OF RANDOM DIGITS*

84807	95930	36761	71148	11800	37663	74799	04145	44546	33163
78780	58545	27269	09229	13242	44294	96587	93720	22795	18092
49336	61953	62074	96231	03453	73259	47849	29464	49702	53699
40323	81992	46550	69692	53509	17708	44358	61969	76589	32677
68431	52129	67042	04001	02626	20015	71126	63844	70618	17821
33211	98245	79689	44640	51129	67337	21933	99058	97429	78638
54542	21028	54091	09809	94902	86125	93030	94031	92484	29999
22363	87396	29784	80123	50946	32774	26930	68782	99184	51393
02134	37278	09876	10384	60989	12563	67880	94330	95079	20544
86623	01483	37535	02599	43260	41937	23696	84361	94916	52832
13588	88282	30806	44904	04844	24192	52799	78036	55671	93997
37687	58873	42290	86952	13142	84776	91263	80551	92162	58172
71321	46526	33025	03492	06236	98869	58737	45989	42135	02388
19992	18431	58335	47967	34287	91198	62765	77604	94945	97880
44011	43099	35247	39278	80532	21337	00727	14614	42651	09416
86263	96354	90566	71538	26263	36357	25121	94918	51486	13434
72285	52257	41496	68568	21713	89619	90463	61741	64754	79313
66314	99626	34046	57643	30290	84472	27138	48894	39547	38356
65186	49962	81768	67441	92145	04381	43270	13919	52671	00178
35798	09155	38747	83376	03746	27242	68272	71519	08184	55800
84704	05094	81787	28634	16609	83064	53882	19778	42642	04293
74107	83335	67860	33969	01881	72484	19576	87797	04791	39049
62631	89062	76786	10489	95535	50315	14058	84899	09260	64472
40986	24967	64478	31362	32544	87057	87782	26762	88003	65274
48550	57975	79790	19843	96658	05293	00368	42522	19868	73137
56490	75689	70233	51120	32269	80131	82056	04828	54197	17092
26467	65708	87113	59280	17852	09297	30655	03999	36912	02460
48037	03246	73165	70959	11459	57358	01662	97116	29005	33167
13646	58293	11038	23565	50681	93772	54008	02783	91449	59266
38961	95851	81091	09049	35544	04269	39758	98970	79902	27467
74069	87734	66384	37587	58970	93451	00883	78210	43118	88787
93155	00861	47558	69357	76741	36831	90331	12629	70651	25709
23224	74317	05620	79839	41415	27579	41763	51799	82166	10765
79284	31960	50429	98444	99915	10488	79114	22904	94075	51249
20377	68019	41055	79458	01913	78996	17378	27284	47096	75931
10386	43403	79209	65679	75371	65250	92122	55911	97908	78311
09104	09820	57496	63450	73584	99354	85254	61105	59574	84289
07944	48624	05305	33996	07289	08004	40261	90150	52809	31367
67058	21255	60672	72823	72416	84196	01762	02728	24101	21949
85965	67992	59634	04891	21512	78936	27666	31118	54031	24841
69084	00415	32583	45049	75659	92758	04574	10732	13608	78019
74761	73257	08330	22770	31205	98039	36766	87547	26619	83307
10798	80895	41309	89056	21846	02064	42997	21426	15119	78523
64972	09793	50166	91085	83796	20652	50989	11419	15357	83281
53946	53079	04800	79780	05384	65615	87837	43354	75244	74011
24108	26649	12973	22013	92017	99207	52069	89184	96279	25993
87328	59395	61133	65119	32199	60177	53397	24368	46271	28055
05131	66287	97594	98851	62888	95450	64560	88038	98366	48618
10170	50559	73587	47263	16334	99229	54374	79264	17934	68791
37088	63192	31714	21916	57498	41448	76994	30005	51196	33160

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